Analysis 1
Colloquium of Week 7
Limit Points and Sequences

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1 Limit Points of a Set

Let (X, d) be a metric space.

1.1 Recall A Few Definitions

1.1.1 Open Balls

Let $x \in X$ and $\varepsilon > 0$. An open ball at x with radius ε, denoted by $B_\varepsilon(x)$, is defined as

$$\boxed{B_\varepsilon(x) := \{ y \in X \mid d(x, y) < \varepsilon \}}$$

1.1.2 Open Sets in a Metric Space

The property on subsets of metric spaces called being open: The set $U \subseteq X$ is open iff $\forall x \in U \exists \varepsilon > 0 B_\varepsilon(x) \subseteq U$. Denote the set of all open sets on X as $Open(X)$. Thus we have

$$\boxed{Open(X) \equiv \{ U \subseteq X \mid \forall x \in U (\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq U) \}}$$

1.2 Open Neighborhouds of a Point

An open neighborhoud of a point $x \in X$ is an open subset $U \in Open(X)$ which contains x: $U \subseteq X$ is an open neighborhoud of x iff $U \in Open(X) \wedge x \in U$.

1.3 Limit Points of a Subset in a Metric Space

- Let $E \subseteq X$. Let $p \in X$.
- *Definition:* p is a limit point of E iff every open neighborhoud U of p contains a point $q \in U \setminus \{p\}$ such that $q \in E$.
  - If you want a more "computerized" statement of the definition: p is a limit point of E iff $(\forall U \in Open(X) : (p \in U)) \exists q \in ((U \setminus \{p\}) \cap E)$.
  - From this definition it follows immediately that
    Claim: If $W \subseteq E$ and p is a limit point of W then p is a limit point of E.
    Proof:
    * Let a nbhd U of p be given.
    * Because p is a limit point of W, then $\exists q \in ((U \setminus \{p\}) \cap W)$.
    * But $W \subseteq E$, so that $q \in E$ as well.
    * Thus $\exists q \in ((U \setminus \{p\}) \cap E)$.

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  - Note: we could also formulate everything with open balls instead. For metric spaces this is equivalent. The reason we insist on formulating things in terms of open sets / open neighborhouds rather than open balls is that next year when you speka about nonmetrizable topologies you will have an easy life. In fact,
    Claim: p is a limit point of $E \iff \forall \varepsilon > 0 \exists q \in (B_\varepsilon(p) \setminus \{p\}) \cap E$.
    Proof:
    * $\boxed{\implies}$
      · Assume p is a limit point of E.
      · Let $\varepsilon > 0$ be given.

· Then $p \in B_\varepsilon(p) \in Open(X)$, that is, the open ball $B_\varepsilon(p)$ around p *is* an open neighbourhood of p.
· As such, we may apply the condition of p being a limit point of E on $B_\varepsilon(p)$ itself as the open neighborhoud to conclude that: $\exists q \in (B_\varepsilon(p) \setminus \{p\}) \cap E$.

* $\boxed{\Longleftarrow}$
  · Assume $\forall \varepsilon > 0 \exists q \in (B_\varepsilon(p) \setminus \{p\}) \cap E$.
  · Let some $U \in Open(X) : p \in U$ be given (that is, let some open neighborhoud of p be given).
  · Because $U \in Open(X)$, $\exists \varepsilon_0 > 0 : B_{\varepsilon_0}(p) \subseteq U$.
  · So apply the assumption on $\varepsilon_0 > 0$, to obtain that $\exists q \in (B_{\varepsilon_0}(p) \setminus \{p\}) \cap E$.
  · But $B_{\varepsilon_0}(p) \subseteq U$, so that $q \in B_{\varepsilon_0}(p) \setminus \{p\}$ means $q \in U \setminus \{p\}$. Of course, we still have $q \in E$.
  · Thus the condition for p being a limit point is fulfilled.

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- Note that $p \in X$ does not have to lie inside of E in order to be a limit point of E (see theorem and example below).
- Note that finite sets have no limit points (make a picture of an open nbhd around p which will clearly avoid any point of a finite set).
- Speaking for the moment about the case where $X = \mathbb{R}$ and $d = d_{Euclidean}$, there is a close connection between limit points of sets, supremums and infimums. In fact,
  Claim: If $\sup(E) \notin E$ then $\sup(E)$ is a limit point of E.
  Proof:
  * We will use the above theorem in order to prove limit points with open balls.
  * So let $\varepsilon_0 > 0$ be given. We need to find that point $q \in (B_\varepsilon(\sup(E)) \setminus \{\sup(E)\}) \cap E$.
  * Remember the approximation property for the supremum: $\forall \varepsilon > 0 \exists x_\varepsilon \in E$ such that $\sup(E) - \varepsilon < x_\varepsilon \leq \sup(E)$.
  * Recall that $\sup(E) \notin E$, and so it is actually possible to refine the above inequalities into $\sup(E) - \varepsilon < x_\varepsilon < \sup(E)$.
  * It will turn out that $x_\varepsilon \in (B_\varepsilon(\sup(E)) \setminus \{\sup(E)\}) \cap E$.
  * We already know that $x_\varepsilon \in E$ by the approximation property.
  * Further, it is known that $\sup(E) - \varepsilon < x_\varepsilon < \sup(E)$, which means that $x_\varepsilon \in B_\varepsilon(\sup(E)) \setminus \{\sup(E)\}$.

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- The very same theorem holds for $\inf(E)$ when $\inf(E) \notin E$, with an analogous proof.
- The above theorem would fail if $\sup(E) \in E$, for instance, take $E = (0, 1) \cup \{2\}$. Then $\sup(E) = 2$, $2 \in E$ and we can find nbhds around 2 which will contain no other members of E.
- The above theorem also doesn't stipulate what happens if $\sup(E) \in E$. It just says it *can* prove that if $\sup(E) \notin E$, then $\sup(E)$ is a limit point of E. However, clearly we can find of situations where $\sup(E) \in E$ and $\sup(E)$ is still a limit point of E: Take $E = [0, 1]$. Then $\sup(E) = 1$ and clearly *every* open nbhd of 1 will contain some points of E.

- Example: Define the set $E := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\} \right\}$ inside of the metric space $(\mathbb{R}, d_{Euclidean})$.
  Claim: $0 \in \mathbb{R}$ is a limit point of E (even though $0 \notin E$!)
  Proof:
  - First show that $\inf(E) = 0$ (I trust you are able to do this).
  - Observe that $0 \notin E$.
  - Use the above theorem.

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2 Limit Points of a Sequence

We are still assuming that (X, d) is a metric space.

- Recall that a sequence is just a map $f : \mathbb{N} \to X$.

2.1 Image of a Map

- Recall the definition of the image of a set under a map, $f(\mathbb{N}) \equiv \{ f(n) \in X \mid n \in \mathbb{N} \}$. When we feed the whole domain set, we call this the image of the map.
- Viewed as a subset of X, $f(\mathbb{N}) \subseteq X$, we may ask, does $f(\mathbb{N})$ have any limit points?

2.2 Limit Point of a Sequence (as opposed to of a Set)

- *Definition*: $p \in X$ is "a limit point of the sequence $(f_n)_{n\in\mathbb{N}}$" iff $p \in X$ is a limit point of the set $f(\mathbb{N})$, where "limit point of a set" is as defined above, and by $f(\mathbb{N})$ we mean the image of $\mathbb{N}$ under the map $f : \mathbb{N} \to X$ corresponding to the sequence $(f_n)_{n\in\mathbb{N}}$.

2.3 Subsequences Attack Again

- Recall from last Friday that a subsequence is defined via a subset $A \subseteq \mathbb{N}$ such that $|A| = |\mathbb{N}|$ (***forgot to mention this on Friday:*** *A* ***cannot be finite!***). Then we say that $(f_n)_{n\in A}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$ and we could prove (or not) that a subsequence converges using the same definition for the whole sequence, only now replacing everywhere $\mathbb{N}$ with A.
- Work with $(X,\, d) = (\mathbb{R},\, d_{Euclidean})$ for simplicity now (to speak about order).
- A subsequence A diverges to infinity iff $\forall M \in \mathbb{R} \exists m_M \in A : ((n \geq m_M \implies f_n > M)\, \forall n \in A)$.

2.4 Lim Sup?

- Recall also the definition of the lim sup and lim inf: We defined, for a given sequence $(f_n)_{n\in\mathbb{N}}$, the set E of all points in $\mathbb{R}$, such that $\exists$ a subsequence A of $(f_n)_{n\in\mathbb{N}}$ which converges to that point:

$$\boxed{E_{(f_n)_{n\in\mathbb{N}}} := \left\{\, x \in X \;\middle|\; \exists A \subseteq \mathbb{N} : (f_n)_{n\in A} \to x \,\right\}}$$

  we also agreed, by convention, that if $(f_n)_{n\in A}$ diverges to infinity or minus infinity, then we include plus or minus infinity in E.

- Then we defined $\limsup_{n\to\infty} f_n := \sup\left(E_{(f_n)_{n\in\mathbb{N}}}\right)$ and $\liminf_{n\to\infty} := \inf\left(E_{(f_n)_{n\in\mathbb{N}}}\right)$.

2.5 Connection Between Limit Point of a Sequence and Subsequential Convergence

- *Claim*: If $(f_n)_{n\in\mathbb{N}}$ is a sequence which is constant with up to finitely many terms, then:
  $x \in \mathbb{R}$ is a limit point of the sequence $(f_n)_{n\in\mathbb{N}} \iff x \in E_{(f_n)_{n\in\mathbb{N}}}$.
  Proof:
  - $\boxed{\implies}$
    * Assume $x \in \mathbb{R}$ is a limit point of $(f_n)_{n\in\mathbb{N}}$.
    * Then let $\varepsilon > 0$ be given.
    * Our goal is to show that there exist some $A \subseteq \mathbb{N}$ and some $m_\varepsilon \in A$ such that $(n \geq m_\varepsilon \implies |f_n - x| < \varepsilon)$ for all $n \in A$, which will show that x is the limit point of the subsequence defined by $A \subseteq \mathbb{N}$.
    * So define A as follows:
      · Because $x \in \mathbb{R}$ is a limit point of $(f_n)_{n\in\mathbb{N}}$, we know that $\exists n_1 \in \mathbb{N}$ such that $f_{n_1} \in B_\varepsilon(x) \setminus \{x\}$, that is $0 < |f_{n_1} - x| < \varepsilon$.
      · Define $\varepsilon_1 := |f_{n_1} - x|$. Thus $0 < \varepsilon_1 < \varepsilon$.
      · Because x is a limit point of $(f_n)_{n\in\mathbb{N}}$, $\exists n_2 \in \mathbb{N} : f_{n_2} \in B_{\varepsilon_1}(x) \setminus \{x\}$.
      · However, we know that $\varepsilon_1 < \varepsilon$. Thus, $f_{n_2} \neq f_{n_1}$. Thus, $n_2 \neq n_1$.
      · Continue in this fashion ad infinitum (we are guaranteed that we can do this by the principle of inductive definition (for details return to Halmos end of chapter 12 on recursive definition)).
      · Define $A := \{n_1,\, n_2,\, n_3,\, \dots\}$.
    * By construction it is clear that A and $n_1 \in A$ obey the definition for the convergence of a subsequence, a subsequence *defined* by A.
  - $\boxed{\impliedby}$
    * Assume $x \in E_{(f_n)_{n\in\mathbb{N}}}$.
    * For the moment assume that $x \neq \pm\infty$ (it is your homework to complete the proof by dealing with this case).
    * Let A be the subsequence corresponding to x, that is $(f_n)_{n\in A} \to x$.
    * Our goal is to show that x is a limit point of the sequence $(f_n)_{n\in\mathbb{N}}$, that is a limit point of the set $f(\mathbb{N})$.
    * To that end, let $\varepsilon > 0$ be given.
    * Because $(f_n)_{n\in A} \to x$, we know that $\exists m_\varepsilon \in A$ such that $(n \geq m_\varepsilon \implies |f_n - x| < \varepsilon)\, \forall n \in A$.
    * If $f_{m_\varepsilon} \neq x$ then $f_{m_\varepsilon} \in B_\varepsilon(x) \setminus \{x\}$ and we are done.
    * Otherwise, because $(f_n)_{n\in\mathbb{N}}$ is constant for only finitely many terms, there must be some $n_0 \in A$ such that $n_0 \geq m_\varepsilon$ and $f_{n_0} \neq x$ (otherwise f_n is a constant (namely x) for infinitely many terms!).
    * So take $f_{n_0} \in B_\varepsilon(x) \setminus \{x\}$.

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- *Claim:* $(f_n)_{n\in\mathbb{N}}$ converges (or diverges to plus or minus infinity) iff $\left|E_{(f_n)_{n\in\mathbb{N}}}\right| = 1$.
  Proof: homework (not hard).

3 Dense Subsets / Dense Sequences

- Let (X, d) be a metric space.
- *Definition*: A subset $E \subseteq X$ is *dense* iff $\forall x \in X$, $x \in E$ or x is a limit point of E.
  - The definition of a dense sequence is obtained by considering the image set of the map corresponding to the sequence, as in the discussion above for the limit point of a sequence.
- *Claim:* $\mathbb{Q}$ is dense in $\mathbb{R}$.
  Proof:
  - Let $x \in \mathbb{R}$ be given.
  - If by luck $x \in \mathbb{Q}$ then we are done, otherwise:
  - Write the decimal expansion of x: we know this involves an infinite sequence, which is actually an infinite series, which converges to x. Each item in this sequence is in fact a rational number (a decimal number with a finite representation) as you have seen in class.
  - Because we found a sequence of rational numbers whose limit is x, we can use the above theorem to conclude that x is a limit point of the image set of that sequence.
  - But the limit point of a subset is a limit point of a larger ambient set, and the image set of that sequence is a subset of $\mathbb{Q}$. Hence x is a limit point of $\mathbb{Q}$.

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3.1 An Example of A Dense Sequence

- Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be given.
- Let $\{r\}$ denote the fractional part of r: $r - \lfloor r \rfloor$ where $\lfloor \xi \rfloor$ denotes the floor function (the smallest positive integer in ξ).
- Define $x_n := \{\alpha n\}$ for all $n \in \mathbb{N}$.
- *Claim*: $(x_n)_{n\in\mathbb{N}}$ is dense in $[0, 1]$.
  Proof[1]*:*
  - *Claim*: If $(i, j) \in \mathbb{N}^2$ such that $i \neq j$, then $\{i\alpha\} \neq \{j\alpha\}$.
    * If this were not true, then
    $$i\alpha - \lfloor i\alpha \rfloor = \{i\alpha\} = \{j\alpha\} = j\alpha - \lfloor j\alpha \rfloor$$
    and so $i\alpha - j\alpha = \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor$ or $(i-j)\,\alpha = \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor$. But then we have $\alpha = \dfrac{\lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor}{i-j}$ and $\dfrac{\lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor}{i-j} \in \mathbb{Q}$ (because the denominator and numerator are both integers!), which means that $\alpha \in \mathbb{Q}$, a contradiction.
  - Hence, $x\,(\mathbb{N})$ is an infinite subset of $[0, 1]$.
  - By the Bolzano-Weierstrass Theorem[2], $x\,(\mathbb{N})$ has a limit point in $[0, 1]$, as $(x_n)_{n\in\mathbb{N}}$ is bounded.
  - One can thus find pairs of elements of $x\,(\mathbb{N})$ that are arbitrarily close, because there is a convergent subsequence!
  - Let $n \in \mathbb{N}$ be given.
  - So there exist distinct $(i, j) \in \mathbb{N}^2$ such that
    $$0 < |\{i\alpha\} - \{j\alpha\}| < \frac{1}{n}.$$
  - WLOG, it may be assumed that $0 < \{i\alpha\} - \{j\alpha\} < \dfrac{1}{n}$.
  - Let M be the largest positive integer such that $M(\{i\alpha\} - \{j\alpha\}) \leq 1$. (We can have $M > n$ by chance.)
  - The irrationality of α then yields
    $$(*) \quad M(\{i\alpha\} - \{j\alpha\}) < 1$$
    (otherwise the fractional part of α is rational!)
  - Next, observe that for any $m \in \{0, \ldots, n-1\}$, we can find a $k \in \{1, \ldots, M\}$ such that
    $$k(\{i\alpha\} - \{j\alpha\}) \in \left[\frac{m}{n}, \frac{m+1}{n}\right]$$
    This is because (the pigeon-hole principle):

[1] *Adapted from http://math.stackexchange.com/a/272713/61151*
[2] each bounded sequence in $\mathbb{R}$ has a convergent subsequence

    * $m < n$, so that $\left[\frac{m}{n}, \frac{m+1}{n}\right] \subseteq [0, 1]$.
    * $k < M$, so that $k(\{i\alpha\} - \{j\alpha\}) < M(\{i\alpha\} - \{j\alpha\}) < 1$, so that $k(\{i\alpha\} - \{j\alpha\}) < 1$.
    * The length of the interval $\left[\dfrac{m}{n}, \dfrac{m+1}{n}\right]$ equals $\dfrac{1}{n}$, while
    * The distance between $l\left(\{i\alpha\} - \{j\alpha\}\right)$ and $(l+1)\left(\{i\alpha\} - \{j\alpha\}\right)$ equals $\{i\alpha\} - \{j\alpha\}$ and $\{i\alpha\} - \{j\alpha\} < \dfrac{1}{n}$. This holds for all $l \in \mathbb{N}$.

- On the other hand, there is another expression for $k(\{i\alpha\} - \{j\alpha\})$:

$$\begin{aligned} k(\{i\alpha\} - \{j\alpha\}) &= \{k(\{i\alpha\} - \{j\alpha\})\} \quad (\text{As } 0 < k(\{i\alpha\} - \{j\alpha\}) < 1; \text{ see } *.) &(1)\\ &= \{k[(i\alpha - \lfloor i\alpha \rfloor) - (j\alpha - \lfloor j\alpha \rfloor)]\} &(2)\\ &= \{k(i-j)\alpha + k(\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor)\} &(3)\\ &= \{k(i-j)\alpha\}. \quad (\text{The } \{\cdot\}\text{function discards any integer part.}) &(4)\end{aligned}$$

- Hence,

$$\{\underbrace{k(i-j)}_{\mathbb{N}}\alpha\} \in \left[\frac{m}{n}, \frac{m+1}{n}\right] \cap x\left(\mathbb{N}\right)$$

- As n is arbitrary, every non-degenerate sub-interval of $[0,1]$ (an interval which is not a singleton), no matter how small, must contain an element of $x\left(\mathbb{N}\right)$.
- Thus, given any element $\beta \in \mathbb{R}$, take an open ball around it, that's a non-degenerate interval, which would contain an element of $x\left(\mathbb{N}\right)$.

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