Analysis 1
Colloquium of Week 8
Space-Filling Curves

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Abstract

We trace the construction of space-filling curves, which are continuous surjective maps from $[0, 1]$ with the Euclidean metric onto $[0, 1]^2$ (also with the Euclidean metric), due to Schoenberg (1938). As reference, see Rudin's *Principles of Mathematical Analysis* Chapter 8 Exercise 14 page 168 (and references therein). The first such construction is actually due to Peano. A description of that curve can be found in Munkres's *Topology* 2nd edition section 44 page 271.

1 Continuous Maps between Metric Spaces

- Recall that if $f \in Y^X$ where both X and Y have their respective metrics d_X and d_Y defined on them, then
- $\boxed{f \text{ is continuous at } x_0 \in X \Longleftrightarrow \forall \varepsilon > 0\ \exists \delta(\varepsilon, x_0)\colon \forall x \in X\colon \{[d_X(x, x_0) < \delta(\varepsilon, x_0)] \Longrightarrow [d_Y(f(x), f(x_0)) < \varepsilon]\}.}$
- If f is continuous at x_0 for all $x_0 \in X$ then f is called simply continuous.

2 Sequences and Series of Functions

- X and Y are metric spaces as before with metrics d_X and d_Y.
- Let $f \in Y^{\mathbb{N} \times X}$ be a family of maps from $X \to Y$, indexed by $\mathbb{N}$. One particular map is denoted by $f_n : X \to Y$.
- Observe that for each $x \in X$, the set of numbers $\{ f_n(x) \mid n \in \mathbb{N} \}$ actually defines a sequence $(f_n(x))_{n\in\mathbb{N}}$.
- Assume that for each $x \in X$, this sequence $(f_n(x))_{n\in\mathbb{N}}$ indeed converges, to some number which we denote as α_x: $(f_n(x))_{n\in\mathbb{N}} \to \alpha_x$.
- Thus this induces a new function $\varphi : X \to Y$, $x \overset{\varphi}{\mapsto} \alpha_x$, which, at any point x, is defined as the limit of $(f_n(x))_{n\in\mathbb{N}}$. This function is well defined due to our ad-hoc assumption that $(f_n(x))_{n\in\mathbb{N}}$ indeed converges for all $x \in X$.
- Under this circumstances, we use a special terminology: we say that $(f_n)_{n\in\mathbb{N}}$ converges *pointwise* to φ. Observe how now, in our notation, the x does not appear. This is because now we are talking about a sequence of *functions* $(f_n)_{n\in\mathbb{N}}$ rather than a sequence of numbers $(f_n(x))_{n\in\mathbb{N}}$, for each $x \in X$.
- In the future you will see that a sequence of functions can converge to a given function in many different "ways", and that *pointwise convergence* is in a way a rather weak type of convergence of functions.
- The first question which you should ask yourself now is what kind of properties of functions carry over to their limit function, and under which circumstances?
  - If each f_n is a continuous function for each $n \in \mathbb{N}$, and $(f_n)_{n\in\mathbb{N}}$ converges *pointwise* to φ, then can we say that φ is also continuous? If not in general, then under which circumstances is it true?

2.1 Series of Functions

- Everything that was mentioned above can also be done by constructing a *series* of functions:
  - Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $X \to Y$.
  - Then for each $x \in X$, $\left(\sum_{j=1}^{n} f_j(x)\right)_{n\in\mathbb{N}}$ defines a series (of numbers, each series for each x).

- Then if $\left(\sum_{j=1}^{n} f_j(x)\right)_{n\in\mathbb{N}}$ converges for each $x \in X$, define a new function $\varphi \in Y^X$ by $x \overset{\varphi}{\mapsto} \sum_{j=1}^{\infty} f_j(x)$.
- We say then that the series of functions $\sum f_n$ converges to φ *pointwise.*

- There really is no difference between sequences and series of functions, other than the fact that in the definition we add the sum operation. Of course, you could always work with sequences if you really wanted to, by first performing the sum and only then talking about the sequence:
  - Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $X \to Y$.
  - Define a new sequence of functions $(g_n)_{n\in\mathbb{N}}$ from $X \to Y$ by $g_n(x) := \sum_{j=1}^{n} f_j(x)$ for all $x \in X$ and for all $n \in \mathbb{N}$.
  - Then apply the formalism of sequence on $(g_n)_{n\in\mathbb{N}}$.

3 Uniform Convergence of a Sequence of Functions

- To answer the question posed above, we investigate another way of convergence, which is stronger than the pointwise convergence defined above.
- But first let us make more concrete what we have defined as uniform convergence:
  - Let (X, d_X) and (Y, d_Y) be two metric spaces and assume that $\forall n \in \mathbb{N},\ f_n \in Y^X$.
  - Assume that $\forall x \in X,\ (f_n(x))_{n\in\mathbb{N}}$ converges to something. This induces a function $\varphi \in Y^X$ defined by

$$\varphi(x) := \lim_{n\to\infty} f_n(x) \quad \forall x \in X$$

  - We say that $(f_n)_{n\in\mathbb{N}} \to \varphi$ pointwise iff
    * $\forall x \in X, \forall \varepsilon > 0 \exists m(\varepsilon, x) \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq m(\varepsilon, x) \Longrightarrow d_Y(f_n(x), \varphi(x))$.
  - Observe how we have actually done nothing new. We merely restated the definiton of the convergence of a sequence. Only now we have a *family* of sequences $(f_n(x))_{n\in\mathbb{N}}$ indexed by $x \in X$.[1]
  - Observe how $\forall x \in X$, this special $m(\varepsilon, x) \in \mathbb{N}$ depends not only on ε, but also on x.
- So uniform convergence is when one is able to find a special number $m(\varepsilon) \in \mathbb{N}$ which satisfies the definition uniformly for all $x \in X$. Formally:
  - We say that $(f_n)_{n\in\mathbb{N}} \to \varphi$ uniformly iff
    * $\forall x \in X, \forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq m(\varepsilon) \Longrightarrow d_Y(f_n(x), \varphi(x))$ where $m(\varepsilon)$ does not depend on $x \in X$.
- So uniform convergence clearly implies pointwise convergence, but the converse is false (there are many notable examples which I won't repeat here: see Rudin beginning of chapter 7). In this sense, uniform convergence is "stronger".

3.1 How to Obtain Uniform Convergence

There are a number of theorems that will help you prove that a sequence of functions convergences *uniformly.* I am only interested to describe here one due to Weirstrass, which we will use in what follows.

- *Claim*: Let $f \in Y^{\mathbb{N}\times X}$ be a sequence of functions such that there exists a sequence of points in $\mathbb{R}$, $(M_n)_{n\in\mathbb{N}}$, with the property that $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$.
  Then $\left(\sum_{j=1}^{n} f_j\right)_{n\in\mathbb{N}}$ converges uniformly on X if $\left(\sum_{j=1}^{n} M_j\right)_{n\in\mathbb{N}}$ converges.
  Proof:
  - See Rudin theorem 7.10 on page 148.

3.2 Uniform Convergence Carries over Continuity

- *Claim*: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on X and if $f_n \to \varphi$ *uniformly* on X then φ is continuous on X.
  Proof:
  - See Theorems 7.11 and 7.12 in Rudin (page 150).

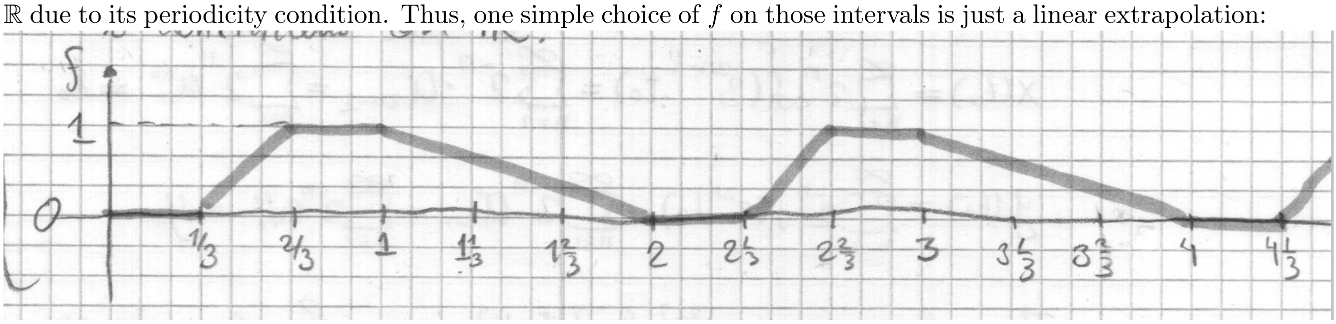
[1] If this confuses you think about the fact that $f \in Y^{\mathbb{N}\times X}$ is actually a function of two parameters, $n \in \mathbb{N}$ and $x \in X$. Then you could view it as a family of functions of *one* parameter (either x or n) indexed by the other parameter (either n or x). When we write $(f_n)_{n\in\mathbb{N}}$ we are thinking about a family of functions $X \to Y$ indexed by n. When write $(f_n(x))_{n\in\mathbb{N}}$ we are thinking of a family of sequences (and recall that sequences are functions $\mathbb{N} \to Y$) indexed by x). The notation may be assymetric but there is definitely a duality. It is not complete because X is a metric space and $\mathbb{N}$ is not.

4 Space-Filling Curves

- For our purposes, we define a Y-filling curve as a continuous map $f \in Y^{[0,\,1]}$ which is both continuous and surjective (we implicity take the Euclidean metric on $[0,\,1]$).
- For this class, we take Y to be the Cartesian product $[0,\,1] \times [0,\,1]$.
- There is actually more than one such map. We will describe a construction due to Schoenberg.

5 Construction of Our Particular Space-Filling Curve

1. Let $f \in [0,\,1]^{\mathbb{R}}$ be a continuous map such that:
   (a) $f(t+2) = f(t)$ for all $t \in \mathbb{R}$.
   (b) $f(t) = 0$ for all $t \in \left[0,\,\frac{1}{3}\right]$.
   (c) $f(t) = 1$ for all $t \in \left[\frac{2}{3},\,1\right]$.
2. Observe that we are *not* stipulating what the value of f should be on the interval $\left(\frac{1}{3},\,\frac{2}{3}\right)$ just so long as f turns out to be continuous. We are also not stipulating what the value of f should be on the interval $(1,\,2)$, again, just so long as f turns out to be continuous. Otherwise, f is completely determined on the whole of $\mathbb{R}$ due to its periodicity condition. Thus, one simple choice of f on those intervals is just a linear extrapolation:



3. It is obvious that such a function is indeed continuous (if it is not obvious to you please prove it).
4. Define two new series of functions, $x \in [0,\,1]^{\mathbb{N}\times[0,\,1]}$ and $y \in [0,\,1]^{\mathbb{N}\times[0,\,1]}$ by the following:
   - $x_n(t) := \sum_{j=1}^{n} 2^{-j} f\left(3^{2j-1}t\right)$ for all $(n,\,t) \in \mathbb{N} \times [0,\,1]$.
   - $y_n(t) := \sum_{j=1}^{n} 2^{-j} f\left(3^{2j}t\right)$ for all $(n,\,t) \in \mathbb{N} \times [0,\,1]$.
5. To see that these functions are well-defined, we need to show that they indeed map to their supposed range (which is clear because f is always between 0 and 1 and so is 2^{-j} where $\sum 2^{-j}$ converges to 1), and that they converge at least pointwise.
6. *Claim:* $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge uniformly on $\mathbb{R}$.
   Proof:
   - Using the above theorem due to Weirstrass, we need to find some sequence $(M_n)_{n\in\mathbb{N}}$ such that $|x_n(t)| \leq M_n$ for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$.
   - Try the sequence $M_n := 2^{-n}$.
   - Because $f \in [0,\,1]$ for any t, $2^{-n} f\left(3^{2n-1}t\right) \leq 2^{-n}$ and so indeed $|x_n(t)| \leq M_n$.
   - But of course $\sum 2^{-n}$ converges to 1: $\lim_{n\to\infty} \sum_{j=1}^{n} 2^{-j} = \frac{2^{-1}}{1-2^{-1}} = 1$.
   - As a result, we may apply the theorem to conclude that $(x_n)_{n\in\mathbb{N}}$ converges uniformly.
   - The same procedure works for $(y_n)_{n\in\mathbb{N}}$ as well.

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7. Of course, for each $n \in \mathbb{N}$, $x_n : \mathbb{R} \to \mathbb{R}$ and $y_n : \mathbb{R} \to \mathbb{R}$ are just sums and products of continuous functions, and as such (using a basic theorem: Rudin 4.10), for each $n \in \mathbb{N}$, these functions are continuous.
8. Then, because they converge uniformly, their limit functions are also continuous:
   - The limit of x_n, $t \mapsto \sum_{j=1}^{\infty} 2^{-j} f\left(3^{2j-1}t\right)$ is continuous. Denote this limit function as $x(t)$.
   - The limit of y_n, $t \mapsto \sum_{j=1}^{\infty} 2^{-j} f\left(3^{2j}t\right)$ is continuous. Denote this limit function as $y(t)$.
9. Next, define a new function $\varphi : [0,\,1] \to [0,\,1]^2$ by $t \mapsto (x(t),\,y(t))$.

10. *Claim*: φ is continuous on $[0,\, 1]$.
Proof:

- What we need to show is that a function defined as a Cartesian product is continuous using the fact that each component function is continuous.
- To that effect the proof is easy and those who are interested can see theorem 4.10 in Rudin.

11. As a result we are *halfway* done because we now have at least a *continuous* map from $[0,\, 1] \to [0,\, 1]^2$. But is it surjective?

12. *Claim*: φ is surjective.
Proof:

- Let $(a,\, b) \in [0,\, 1]^2$ be given.
- Write binary representations of a and b: $a = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}$ and $b = \sum_{n=1}^{\infty} \frac{\beta_n}{2^n}$ where we already know that we can find such sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ where $(\alpha_n,\, \beta_n) \in \{0,\, 1\}^2$ for all $n \in \mathbb{N}$ which will reproduce the numbers a and b (but these sequences are not unique, as we know, because, for example, $(0.0111111\dots)_2 = (0.1)_2$.
- Now build a new sequence, $(\gamma_n)_{n\in\mathbb{N}}$ by the following rule $\gamma_n := \begin{cases} \alpha_{\frac{n+1}{2}} & n \in 2\mathbb{N}+1 \\ \beta_{\frac{n}{2}} & n \in 2\mathbb{N} \end{cases}$. That is, $\{\, \gamma_n \,\} = \{\, \alpha_1,\, \beta_1,\, \alpha_2,\, \beta_2,\, \dots \}$.
- Now define a new number, which is the meaning of the sequence of digits $(\gamma_n)_{n\in\mathbb{N}}$ in ternary representation: $c := \sum_{j=1}^{\infty} \frac{2\gamma_j}{3^{j+1}} = \frac{2\alpha_1}{3^2} + \frac{2\beta_1}{3^3} + \frac{2\alpha_2}{3} + \dots$
- *Claim*: $f\left(3^k c\right) = \gamma_k$ for all $k \in \mathbb{N}$.
Proof:
  - Let $k \in \mathbb{N}$ be given.
  - Make the calculation

$$
\begin{aligned}
f\left(3^k c\right) &\equiv f\left(3^k \sum_{j=1}^{\infty} \frac{2\gamma_j}{3^{j+1}}\right) \\
&= f\left(\sum_{j=1}^{\infty} \frac{2\gamma_j}{3^{j+1-k}}\right) \\
&= f\left(2 \underbrace{\sum_{j=1}^{k-1} 3^{k-j-1}\gamma_j}_{\text{some integer because } k\geq j+1} + \sum_{j=k}^{\infty} \frac{2\gamma_j}{3^{j+1-k}}\right) \\
&\overset{f(2n+t)=f(t)}{=} f\left(\sum_{j=k}^{\infty} \frac{2\gamma_j}{3^{j+1-k}}\right) \\
&= f\left(\underbrace{\frac{2\gamma_k}{3}}_{\in\{0,\, \frac{2}{3}\} \text{ depending on } \gamma_k} + \underbrace{\sum_{j=k+1}^{\infty} \frac{2\gamma_j}{3^{j+1-k}}}_{<\frac{1}{3} \text{ bcs. starts from } \frac{2\gamma_{k+1}}{3^2}+\dots=(0.0(2\gamma_k)\dots)_3}\right) \\
&= \begin{cases} 0 & \gamma_k = 0 \\ 1 & \gamma_k = 1 \end{cases} \\
&= \gamma_k
\end{aligned}
$$

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- As a result we have:

– For $x(c)$:

$$\begin{aligned} x(c) &\equiv \sum_{n=1}^{\infty} 2^{-n} f\left(3^{2n-1}c\right) \\ &= \sum_{n=1}^{\infty} 2^{-n}\gamma_{2n-1} \\ &= \sum_{n=1}^{\infty} 2^{-n}\alpha_n \\ &= a \end{aligned}$$

and similarly for $y(c)$

$$\begin{aligned} y(c) &= \sum_{n=1}^{\infty} 2^{-n} f\left(3^{2n}c\right) \\ &= \sum_{n=1}^{\infty} 2^{-n}\gamma_{2n} \\ &= \sum_{n=1}^{\infty} 2^{-n}\beta_n \\ &= b \end{aligned}$$

- As a result we find that $\varphi(c) = (a, b)$, and so φ is indeed surjective.

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13. Note that in fact, $c \equiv \sum_{j=1}^{\infty} \frac{2\gamma_j}{3^{j+1}}$ which we found for these arbitrary (a, b) is in the Cantor set (because it has only 0 and 2 in its ternary representation: recall the colloquium from week 4). This fact is quite incredible: not only is φ able to cover the whole of the square $[0, 1]^2$, but actually it does so by using only the Cantor set as its domain (which, as we've seen, has measure zero, because it contains almost nothing! (yet its has the Cardinality of the continuum $|\mathbb{R}|$)).

14. The fact that the binary representation is not unique exactly shows why the map φ is not injective.