Analysis 1
Colloquium of Week 9
Continuity, Continuous Extensions, and the Pasting Lemma

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Abstract

We follow some examples from Spivak's *Calculus* (4th edition), present a lemma from Munkres' *Topology* (2nd edition) and show another example of continuous extensions (for more on that see Rudin's *Principles of Mathematical Analysis* chapter 4 exercise 5 (pp. 99)).

1 Some Examples for Continuity

- Define $f:\mathbb{R}\to\mathbb{R}$ by the following rule $x\mapsto\begin{cases}x & x\in\mathbb{Q}\\ 0 & x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}$.
  - *Claim*: f is continuous at 0.
    Proof:
    * Let $\varepsilon>0$ be given.
    * We are looking for a neighborhood of 0, which we denote by $\delta(\varepsilon)>0$, such that $f\left(B_{\delta(\varepsilon)}(0)\right)\subseteq B_{\varepsilon}(f(0))$.
    * Translating this into more "readable" notation, that would mean that if $x\in\mathbb{R}$ is such that $|x|<\delta(\varepsilon)$ (meaning $x\in B_{\delta(\varepsilon)}(0)$), then $|f(x)-f(0)|<\varepsilon$ (meaning $f(x)\in B_{\varepsilon}(f(0))$).
    * Now we should start using the actual definition of f.
    * $f(0)=0$ as $0\in\mathbb{Q}$ and on $\mathbb{Q}$, f is the identity function (sends $x\mapsto x$).
    * So our conditions are that there should be some $\delta(\varepsilon)>0$ such that if $x\in\mathbb{R}$ obeys $|x|<\delta(\varepsilon)$ then $|f(x)|<\varepsilon$.
    * So simply take $\delta(\varepsilon):=\varepsilon$. Why does this work?
      · Divide to two cases:
      1. If $x\in\mathbb{Q}$ then $f(x)=x$ and then since $|x|<\underbrace{\delta(\varepsilon)}_{\varepsilon}$, of course $\underbrace{|f(x)|}_{|x|}<\varepsilon$.
      2. If $x\in\mathbb{R}\backslash\mathbb{Q}$, then $f(x)=0$ and then no matter what $\delta(\varepsilon)$ was chosen to be, $|0|<\varepsilon$.

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  - *Claim*: f is not continuous at x for all $x\in\mathbb{R}\backslash\{0\}$.
    Proof:
    * Let some $x\in\mathbb{R}\backslash\{0\}$ be given.
    * We need to find some $\varepsilon_0>0$ such that no matter which $\delta>0$ we pick, there will always be a point $y\in B_\delta(x)$ which has $f(y)\notin B_{\varepsilon_0}(f(x))$.
    * Case 1: If $x\in\mathbb{Q}$,
      · then $f(x)=x$ and then simply take $\varepsilon_0:=\frac{1}{2}|x|$.
      · No matter how close we get to x (how small $\delta>0$ we pick), that interval around x will always contain some irrational point $y\in B_\delta(x)\backslash\mathbb{Q}$. That irrational point is then arbitrarily close to x, however, its image is 0, and 0 is too far away to be in $B_{\frac{1}{2}|x|}(x)$ [draw picture of the line].
    * Case 2: If $x\notin\mathbb{Q}$,
      · Then $f(x)=0$. Take again $\varepsilon_0:=\frac{1}{2}|x|$.
      · Then let $\delta>0$ be given (we need to show this breaks down for *every* $\delta>0$).
      · So we can always find some rational $y\in B_{\min(\frac{1}{4}|x|,\,\delta)}(x)\cap\mathbb{Q}$ which is sufficiently close to x, and then we will have $|x-y|<\frac{1}{4}|x|$ which implies $||x|-|y||<\frac{1}{4}|x|$ which implies $|y|>\frac{3}{4}|x|$.
      · Then $|f(x)-f(y)|=|f(y)|=|y|>\frac{3}{4}|x|>\frac{1}{2}|x|=\varepsilon_0$.

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- Define $f:\mathbb{R}\to\mathbb{R}$ by the rule $x\mapsto\begin{cases}0 & x\in\mathbb{R}\backslash\mathbb{Q}\\ \frac{1}{q} & x=\frac{p}{q}\wedge p\in\mathbb{Z}\wedge q\in\mathbb{N}\backslash\{0\}\wedge\gcd(p,\,q)=1\end{cases}$.
  - *Claim*: f is continuous at x for all $x\in\mathbb{R}\backslash\mathbb{Q}$.
    Proof: homework.
  - *Claim*: f is not continuous at x for all $x\in\mathbb{Q}$.
    Proof: homework.
- *Claim*: $\exists f\in\mathbb{R}^{\mathbb{R}}$ such that f is not continuous at x for all $x\in\mathbb{R}$ yet $|f|$ (viewed as a new function $\mathbb{R}\to\mathbb{R}$ by the rule $x\mapsto|f(x)|$ for all $x\in\mathbb{R}$) is continuous for all $x\in\mathbb{R}$.
  Proof: homework.
- *Claim*: $f:\mathbb{C}\to\mathbb{C}$ defined by $z\mapsto 2\overline{z}$ is continuous at z for all $z\in\mathbb{C}$.
  Proof:
  - Pick some $z\in\mathbb{C}$.
  - Let $\varepsilon>0$ be given.
  - Take $\delta(\varepsilon,\,z):=\frac{1}{2}\varepsilon$.
  - Then $z'\in B_{\delta(\varepsilon,\,z)}(z)$ implies $|z-z'|<\underbrace{\delta(\varepsilon,\,z)}_{\frac{1}{2}\varepsilon}$.
  - Then

$$\begin{aligned}|f(z)-f(z')| &= \left|2\overline{z}-2\overline{z'}\right|\\ &= 2\left|\overline{z}-\overline{z'}\right|\\ &= 2\left|\overline{z-z'}\right|\\ &= 2|z-z'|\\ &\leq 2\frac{1}{2}\varepsilon\end{aligned}$$

    so we are in business.

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2 Continuous Extensions

- See example from Recitation session of week 8 about continuous extensions.

3 The Pasting Lemma

3.1 A Reminder

- Recall that in the most general definition (the one that transcends metric spaces and with which you shall graduate your degree!) $f:X\to Y$ is continuous iff $f^{-1}(V)\in Open(X)$ for all $V\in Open(Y)$.
- Recall that $Closed(X)\equiv\{\,F\subseteq X\mid X\backslash F\in Open(X)\,\}$.

3.2 Subspace Topology

- Whatever $Open(X)$ was defined as (we have defined it only for metric spaces. There is a more general definition, which is called a topology), given some $A\subseteq X$, we may define $Open(A)$ as:

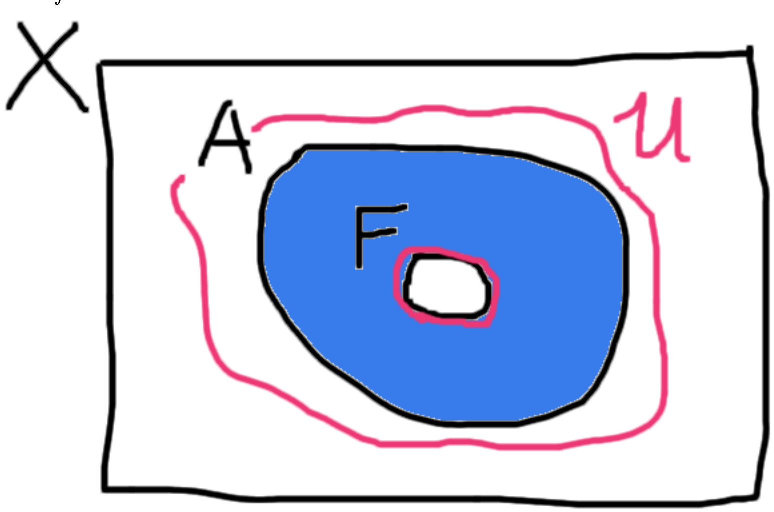
$$Open(A):=\{\,U\subseteq A\mid\exists V\in Open(X)\wedge U=V\cap A\,\}$$

  This is called the “subspace topology”.

3.3 The Actual Pasting Lemma

- Let X and Y be a metric spaces.
- Let $(A,\,B)\in Closed(X)^2$ and assume further that $X=A\cup B$.
- Assume that we have two functions $f:A\to Y$ and $g:B\to Y$.
- Assume that $f(x)=g(x)\,\forall x\in A\cap B$.

- Define a new function, $h : X \to Y$ by $x \overset{h}{\mapsto} \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$ (if $x \in A \cap B$ then we have no ambiguity *by assumption*).
- *Claim*: If both f and g are continuous then h is continuous.
  Proof:
  - *Claim*: $h : X \to Y$ is continuous iff $\forall F \in Closed(Y)$, $h^{-1}(F) \in Closed(X)$.
    Proof:
    * Assume that h is continuous.
      · Let $F \in Closed(Y)$ be given.
      · Then $Y \backslash F \in Open(Y)$.
      · By the continuity of h we have that $h^{-1}(Y \backslash F) \in Open(X)$.
      · However, as we know, the inverse image respects complements, and so $h^{-1}(Y \backslash F) = h^{-1}(Y) \backslash h^{-1}(F)$.
      · But $h^{-1}(Y) = X$.
      · Thus we have that $X \backslash h^{-1}(F) \in Open(X)$, which implies $h^{-1}(F) \in Closed(X)$ as desired.
    * Assume that $\forall F \in Closed(Y)$, $h^{-1}(F) \in Closed(X)$.
      · Let $U \in Open(Y)$ be given.
      · Then $Y \backslash U \in Closed(Y)$.
      · Then by the assumption, $h^{-1}(Y \backslash U) \in Closed(X)$.
      · But as we've seen that means that $X \backslash h^{-1}(U) \in Closed(X)$, or $h^{-1}(U) \in Open(X)$ and so h is continuous.
  - Let $F \in Closed(Y)$ be given.
  - *Claim*: $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$.
    Proof:
    * $\boxed{\subseteq}$
      · Let $x \in h^{-1}(F)$ be given.
      · That implies that $h(x) \in F$.
      · Case 1: $x \in A \backslash B$. Then $h(x) = f(x)$ and so we have that $f(x) \in F$. This in turn implies that $x \in f^{-1}(F)$.
      · Case 2: $x \in B \backslash A$. The same logic implies that $x \in g^{-1}(F)$.
      · Case 3: $x \in A \cap B$. Then $h(x) = f(x) = g(x)$ and then $f(x) \in F$ *and* $g(x) \in F$ which implies that $x \in f^{-1}(F) \cap g^{-1}(F)$.
      · In either case, we have that $x \in f^{-1}(F) \cup g^{-1}(F)$.
    * $\boxed{\supseteq}$
      · Let $x \in \left[f^{-1}(F) \cup g^{-1}(F)\right]$ be given.
      · Case 1: $x \in A \backslash B$. Then either $f(x) \in F$ or $g(x) \in F$.
      1. If $f(x) \in F$, then due to $x \in A$ we have $f(x) = h(x)$ and so $h(x) \in F$ and so $x \in h^{-1}(F)$.
      2. If $g(x) \in F$, then due to $x \in A$, we must have that $x \in A \cap B$ and so again $h(x) \in F$ or $x \in h^{-1}(F)$.
    1. The other cases follow similarly.
  - Because f and g are continuous, and $F \in Closed(Y)$, $f^{-1}(F) \in Closed(A)$ and $g^{-1}(F) \in Closed(B)$.
  - *Claim*: If $A \in Closed(X)$ and $F \in Closed(A)$ then $F \in Closed(X)$.
    Proof:



    * $A \in Closed(X)$ implies that $X \backslash A \in Open(X)$.

    * $F \in Closed(A)$ implies that $A \backslash F \in Open(A)$.
    * But $A \backslash F \in Open(A)$ implies that $A \backslash F = U \cap A$ for some $U \in Open(X)$.
    * But due to $F \subseteq A \subseteq X$ and the fact that unions of open sets are again open, we have:

$$\begin{aligned} X \backslash F &= (A \backslash F) \cup (X \backslash A) \\ &= (U \cap A) \cup (X \backslash A) \\ &\overset{HW1Q2(d)}{=} ((X \backslash A) \cup U) \cap \left(\underbrace{(X \backslash A) \cup A}_{X} \right) \\ &= \underbrace{(X \backslash A)}_{\in Open(X)} \cup \underbrace{U}_{\in Open(X)} \\ &\in Open(X) \end{aligned}$$

    * That is, $X \backslash F \in Open(X)$.
    * Thus $F \in Closed(X)$.
  - As a result, we have $f^{-1}(F) \in Closed(X)$ and $g^{-1}(F) \in Closed(X)$.
  - But then *finite* union of closed sets is again closed, that is, $\left[f^{-1}(F) \cup g^{-1}(F)\right] \in Closed(X)$ or $h^{-1}(F) \in Closed(X)$.
  - Because $F \in Closed(X)$ was arbitrary, we conclude that h is continuous as it fulfills our (new) criteria for continuity.

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- *Claim*: If h as given above is continuous then so are f and g.
  Proof:
  - First we prove an auxiliary result:
  - *Claim*: Let $\alpha : X \to Y$ be continuous and let $A \subseteq X$. Then the new function $\alpha|_A$ defined as $\alpha|_A : A \to Y$ by the rule $x \overset{\alpha|_A}{\mapsto} \alpha(x)$ for all $x \in A$ is continuous.
    Proof:
    * *Claim*: $\alpha|_A{}^{-1}(U) = a^{-1}(U) \cap A$ for all $U \subseteq Y$.
      Proof: homework.
    * Let $U \in Open(Y)$ be given.
    * Then $\alpha|_A{}^{-1}(U) = \alpha^{-1}(U) \cap A$.
    * By the definition of $Open(A) \equiv \{ U \subseteq A \mid \exists V \in Open(X) \wedge U = V \cap A \}$ and the continuity of α (which implies $\alpha^{-1}(U) \in Open(X)$) we have that $\alpha|_A{}^{-1}(U) \in Open(A)$.
  - Because $f = h|_A$ and $g = h|_B$ it follows immediately that f and g are continuous.

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