Analysis 1
Colloquium of Week 9
Continuity, Continuous Extensions, and the Pasting Lemma

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Abstract

We follow some examples from Spivak’s Calculus (4th edition), present a lemma from Munkres’ Topology (2nd edition) and show another example of continuous extensions (for more on that see Rudin’s Principles of Mathematical Analysis chapter 4 exercise 5 (pp. 99)).

1 Some Examples for Continuity

• Define $f : \mathbb{R} \to \mathbb{R}$ by the following rule $x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

  - Claim: $f$ is continuous at $0$.
   
   Proof:
   * Let $\varepsilon > 0$ be given.
   * We are looking for a neighborhood of 0, which we denote by $\delta(\varepsilon) > 0$, such that $f(B_{\delta(\varepsilon)}(0)) \subseteq B_{\varepsilon} \{f(0)\}$.
   * Translating this into more “readable” notation, that would mean that if $x \in \mathbb{R}$ is such that $|x| < \delta(\varepsilon)$ (meaning $x \in B_{\delta(\varepsilon)}(0)$), then $|f(x) - f(0)| < \varepsilon$ (meaning $f(x) \in B_{\varepsilon}(f(0))$).
   * Now we should start using the actual definition of $f$.
   * $f(0) = 0$ as $0 \in \mathbb{Q}$ and on $\mathbb{Q}$, $f$ is the identity function (sends $x \mapsto x$).
   * So our conditions are that there should be some $\delta(\varepsilon) > 0$ such that if $x \in \mathbb{R}$ obeys $|x| < \delta(\varepsilon)$ then $|f(x)| < \varepsilon$.
   * So simply take $\delta(\varepsilon) := \varepsilon$. Why does this work?
     * Divide into two cases:
       1. If $x \in \mathbb{Q}$ then $f(x) = x$ and then since $|x| < \delta(\varepsilon)$, of course $|f(x)| < \varepsilon$.
       2. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x) = 0$ and then no matter what $\delta(\varepsilon)$ was chosen to be, $|0| < \varepsilon$.

  \[\square\]

  - Claim: $f$ is not continuous at $x$ for all $x \in \mathbb{R} \setminus \{0\}$.
   
   Proof:
   * Let some $x \in \mathbb{R} \setminus \{0\}$ be given.
   * We need to find some $\varepsilon_0 > 0$ such that no matter which $\delta > 0$ we pick, there will always be a point $y \in B_\delta(x)$ which has $f(y) \notin B_{\varepsilon_0} \{f(x)\}$.
   * Case 1: If $x \in \mathbb{Q}$,
     * then $f(x) = x$ and then simply take $\varepsilon_0 := \frac{1}{2} |x|$. 
     * No matter how close we get to $x$ (how small $\delta > 0$ we pick), that interval around $x$ will always contain some irrational point $y \in B_\delta (x) \setminus \mathbb{Q}$. That irrational point is then arbitrarily close to $x$, however, its image is 0, and 0 is too far away to be in $B_{\frac{1}{2} |x|} (x)$ [draw picture of the line].
   * Case 2: If $x \notin \mathbb{Q}$,
     * Then $f(x) = 0$. Take again $\varepsilon_0 := \frac{1}{2} |x|$.
     * Then let $\delta > 0$ be given (we need to show this breaks down for every $\delta > 0$).
     * So we can always find some rational $y \in B_{\min \{ \frac{1}{4} |x|, \delta \}}(x) \cap \mathbb{Q}$ which is sufficiently close to $x$, and then we will have $|x - y| < \frac{1}{2} |x|$ which implies $||x| - |y|| < \frac{1}{4} |x|$ which implies $|y| > \frac{3}{4} |x|$.
     * Then $|f(x) - f(y)| = |f(y)| = |y| > \frac{3}{4} |x| > \frac{1}{2} |x| = \varepsilon_0$.

  \[\square\]
• Define \( f : \mathbb{R} \to \mathbb{R} \) by the rule \( x \mapsto \begin{cases} 0 & x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{p}{q} & x = \frac{p}{q} \land p \in \mathbb{Z} \land q \in \mathbb{N}\setminus\{0\} \land \gcd(p, q) = 1 \end{cases} \).

  - Claim: \( f \) is continuous at \( x \) for all \( x \in \mathbb{R}\setminus\mathbb{Q} \).
    Proof: homework.
  - Claim: \( f \) is not continuous at \( x \) for all \( x \in \mathbb{Q} \).
    Proof: homework.

• Claim: \( \exists f \in \mathbb{R}^\mathbb{R} \) such that \( f \) is not continuous at \( x \) for all \( x \in \mathbb{R} \) yet \( |f| \) (viewed as a new function \( \mathbb{R} \to \mathbb{R} \) by the rule \( x \mapsto |f(x)| \) for all \( x \in \mathbb{R} \)) is continuous for all \( x \in \mathbb{R} \).
  Proof: homework.

• Claim: \( f : \mathbb{C} \to \mathbb{C} \) defined by \( z \mapsto 2\pi \) is continuous at \( z \) for all \( z \in \mathbb{C} \).
  Proof:
  - Pick some \( z \in \mathbb{C} \).
  - Let \( \varepsilon > 0 \) be given.
  - Take \( \delta(\varepsilon, z) := \frac{1}{2}\varepsilon \).
  - Then \( z' \in B_{\delta(\varepsilon, z)}(z) \) implies \( |z - z'| < \delta(\varepsilon, z) \).
  - Then
    \[
    |f(z) - f(z')| = |2\pi - 2\pi'|
    = 2|\pi - \pi'|
    = 2|z - z'|
    \leq 2\varepsilon
    \]
    so we are in business.

\[\blacksquare\]

2 Continuous Extensions
• See example from Recitation session of week 8 about continuous extensions.

3 The Pasting Lemma
3.1 A Reminder
• Recall that in the most general definition (the one that transcends metric spaces and with which you shall graduate your degree!) \( f : X \to Y \) is continuous iff \( f^{-1}(V) \in \text{Open}(X) \) for all \( V \in \text{Open}(Y) \).
• Recall that \( \text{Closed}(X) \equiv \{ F \subseteq X \mid X \setminus F \in \text{Open}(X) \} \).

3.2 Subspace Topology
• Whatever \( \text{Open}(X) \) was defined as (we have defined it only for metric spaces. There is a more general definition, which is called a topology), given some \( A \subseteq X \), we may define \( \text{Open}(A) \) as:
  \[
  \text{Open}(A) := \{ U \subseteq A \mid \exists V \in \text{Open}(X) \land U = V \cap A \}
  \]
This is called the “subspace topology”.

3.3 The Actual Pasting Lemma
• Let \( X \) and \( Y \) be a metric spaces.
• Let \( (A, B) \in \text{Closed}(X)^2 \) and assume further that \( X = A \cup B \).
• Assume that we have two functions \( f : A \to Y \) and \( g : B \to Y \).
• Assume that \( f(x) = g(x) \forall x \in A \cap B \).
Claim: If both \( f \) and \( g \) are continuous then \( h \) is continuous.

**Proof:**

- **Claim:** \( h : X \to Y \) is continuous iff \( \forall F \in \text{Closed}(Y), \ h^{-1}(F) \in \text{Closed}(X) \).

  **Proof:**
  
  * Assume that \( h \) is continuous.
    
    - Let \( F \in \text{Closed}(Y) \) be given.
    - Then \( Y \setminus F \in \text{Open}(Y) \).
    - By the continuity of \( h \) we have that \( h^{-1}(Y \setminus F) \in \text{Open}(X) \).
    - However, as we know, the inverse image respects complements, and so \( h^{-1}(Y \setminus F) = h^{-1}(Y) \setminus h^{-1}(F) \).
    - But \( h^{-1}(Y) = X \).
    - Thus we have that \( X \setminus h^{-1}(F) \in \text{Open}(X) \), which implies \( h^{-1}(F) \in \text{Closed}(X) \) as desired.

  * Assume that \( \forall F \in \text{Closed}(Y), \ h^{-1}(F) \in \text{Closed}(X) \).
    
    - Let \( U \in \text{Open}(Y) \) be given.
    - Then \( Y \setminus U \in \text{Closed}(Y) \).
    - Then by the assumption, \( h^{-1}(Y \setminus U) \in \text{Closed}(X) \).
    - But as we’ve seen that means that \( X \setminus h^{-1}(U) \in \text{Closed}(X) \), or \( h^{-1}(U) \in \text{Open}(X) \) and so \( h \) is continuous.

- Let \( F \in \text{Closed}(Y) \) be given.

- **Claim:** \( h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) \).

  **Proof:**
  
  * \( \subseteq \)
    
    - Let \( x \in h^{-1}(F) \) be given.
    - That implies that \( h(x) \in F \).
    - Case 1: \( x \in A \setminus B \). Then \( h(x) = f(x) \) and so we have that \( f(x) \in F \). This in turn implies that \( x \in f^{-1}(F) \).
    - Case 2: \( x \in B \setminus A \). The same logic implies that \( x \in g^{-1}(F) \).
    - Case 3: \( x \in A \cap B \). Then \( h(x) = f(x) = g(x) \) and then \( f(x) \in F \) and \( g(x) \in F \) which implies that \( x \in f^{-1}(F) \cap g^{-1}(F) \).
    - In either case, we have that \( x \in f^{-1}(F) \cup g^{-1}(F) \).

  * \( \supseteq \)
    
    - Let \( x \in [f^{-1}(F) \cup g^{-1}(F)] \) be given.
    - Case 1: \( x \in A \setminus B \). Then either \( f(x) \in F \) or \( g(x) \in F \).
      1. If \( f(x) \in F \), then due to \( x \in A \) we have \( f(x) = h(x) \) and so \( h(x) \in F \) and so \( x \in h^{-1}(F) \).
      2. If \( g(x) \in F \), then due to \( x \in A \), we must have that \( x \in A \cap B \) and so again \( h(x) \in F \) or \( x \in h^{-1}(F) \).
    - 1. The other cases follow similarly.

- Because \( f \) and \( g \) are continuous, and \( F \in \text{Closed}(Y) \), \( f^{-1}(F) \in \text{Closed}(A) \) and \( g^{-1}(F) \in \text{Closed}(B) \).

- **Claim:** If \( A \in \text{Closed}(X) \) and \( F \in \text{Closed}(A) \) then \( F \in \text{Closed}(X) \).

  **Proof:**
  
  * \( A \in \text{Closed}(X) \) implies that \( X \setminus A \in \text{Open}(X) \).
* $F \in \text{Closed} (A)$ implies that $A \setminus F \in \text{Open} (A)$.
* But $A \setminus F \in \text{Open} (A)$ implies that $A \setminus F = U \cap A$ for some $U \in \text{Open} (X)$.
* But due to $F \subseteq A \subseteq X$ and the fact that unions of open sets are again open, we have:

\[
X \setminus F = (A \setminus F) \cup (X \setminus A) = (U \cap A) \cup (X \setminus A)
\]

\[
\overset{HW1Q2(d)}{=} ((X \setminus A) \cup U) \cap \left( \frac{(X \setminus A) \cup A}{X} \right)
\]

\[
= \bigcup_{U \in \text{Open}(X)} \left( X \setminus A \right) \cup U \in \text{Open}(X)
\]

* That is, $X \setminus F \in \text{Open} (X)$.
* Thus $F \in \text{Closed} (X)$.

– As a result, we have $f^{-1} (F) \in \text{Closed} (X)$ and $g^{-1} (F) \in \text{Closed} (X)$.
– But then finite union of closed sets is again closed, that is, $[f^{-1} (F) \cup g^{-1} (F)] \in \text{Closed} (X)$ or $h^{-1} (F) \in \text{Closed} (X)$.
– Because $F \in \text{Closed} (X)$ was arbitrary, we conclude that $h$ is continuous as it fulfills our (new) criteria for continuity.

\[\blacksquare\]

* Claim: If $h$ as given above is continuous then so are $f$ and $g$.

Proof:

– First we prove an auxiliary result:

– Claim: Let $\alpha : X \rightarrow Y$ be continuous and let $A \subseteq X$. Then the new function $\alpha|_A$ defined as $\alpha|_A : A \rightarrow Y$ by the rule $x \mapsto \alpha (x)$ for all $x \in A$ is continuous.

Proof:

* Claim: $\alpha|_A^{-1} (U) = a^{-1} (U) \cap A$ for all $U \subseteq Y$.

Proof: homework.

* Let $U \in \text{Open} (Y)$ be given.

* Then $\alpha|_A^{-1} (U) = a^{-1} (U) \cap A$.

* By the definition of $\text{Open} (A) \equiv \{ U \subseteq A \mid \exists V \in \text{Open} (X) \wedge U = V \cap A \}$ and the continuity of $\alpha$ (which implies $a^{-1} (U) \in \text{Open} (X)$) we have that $\alpha|_A^{-1} (U) \in \text{Open} (A)$.

– Because $f = h|_A$ and $g = h|_B$ it follows immediately that $f$ and $g$ are continuous.

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