Analysis 1 - Exercise of Week 1

September 19, 2014

1 Logistics

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- Course website is at http://www.math.ethz.ch/education/bachelor/lectures/hs2014/math/analysis1

- There will be two weekly hours of exercise on Fridays 8-10 and one weekly hour of colloquium on Wednesdays 15-16 (starting next week on 24/09/2014).

- The exercise and colloquium are meant as a contrast to the lecture, where you can actually ask questions and be active. This is possible because there is no material to go through here: all the material you need to know is given in the lecture, and this small group time is where you deal with it. Thus, the best use of it is for you is to be as active as possible: ask as many questions as you want, verify things that were unclear in the lecture, and get practice in talking in front of people about mathematics.

- Despite the above, in order to facilitate your activeness, there is a very rough and unimportant skeleton to small group lessons:

  - In the exercise classes I will repeat material which is needed to solve the homework, and re-repeat the material that was not understood from the homework that was submitted in the previous week. This will mainly be by way of practical examples. I will also gladly answer any questions you have at any points–so please don't hesitate to interrupt me at any point. Especially welcome are questions about the forecoming homework exercises, including request for tips.
  - In the colloqium we will speak about "fun" topics that are not necessarily mentioned in the lecture and might enrich your knowledge. The next colloqium will deal with the Peano Axioms, and industrious students can already read the book P. R. Halmos *Naive Set Theory* pages 1-49.

- Submit homework on a weekly basis. You may submit HW by Friday 10am either in exercise class or in mailboxes in HG in the lobby of F27, with a box hopefully bearing my name saying “analysis 1”. If you submit HW after this time it will not be graded. You will receive your graded work the following week in class. If you choose to submit your homework, please try to follow the same style of proofs that is presented here (claim/proof hierarchies).

2 Discussion of Material in Homework No. 1

2.1 Natural Induction (Vollstaendige Induktion)

2.1.1 Notation

- $\mathbb{N} \equiv \{1, 2, 3, 4, \ldots\}$. Called the “natural numbers” (Natürliche Zahlen). Sometimes starts from 0.
- Curly brackets (Klammern) denote the enumeration (Aufzählung) of a set (a collection of elements–naive axiomatic definition) by enumerating its elements one by one. Dots invite you to imagine continuing the same process without having to write everything.

2.1.2 What is it about? (Worum geht es?)

- Certain statements are always true: “water is wet.” (Wasser ist nass)
- Other statements are true but have a “parameter” in them: “Humans have 10 fingers.” (”Menschen haben zehn Finger”) Here the parameter is ’10’ and the statement is true only when we use the parameter ’10’ (for most people).
- Yet other statements which have a parameter are true not for a single value of the parameter, but for a whole *range* (Interval) of parameters: “There are humans who have lived to m years.” (”Es gibt Menschen die leben m Jahren”) – This statement is true for many values of m: 1, 50, 75, 100. Probably not for 1000.
- Denote the statement we are dealing with by S. To convey that this statement depends on a parameter, write $S(m)$.
- Some statements are true for any value of m that we choose from $\mathbb{N} \equiv \{1, 2, 3, 4, \ldots\}$. For example, the statement $S(m) =$”$m > -1$” is true regardless of which m we choose, just as long as it is chosen from $\mathbb{N}$:

$$\begin{array}{ccc} 1 & > & -1 \\ 2 & > & -1 \\ 3 & > & -1 \\ & \ldots & \end{array}$$

- We see intuitively that $S(m)$ is true for all m chosen from $\mathbb{N}$, but how do we "prove" this? Induction.

Notation

- In true/false statements, it is conventional to replace the words ("Wir benutzen Abkurzungen")
  - "for all" with the graphical symbol $\forall$ "fuer all"
  - "chosen from" with $\in$ "gewaehlt aus"
  - "and" with $\wedge$. "und"
  - "or" with $\vee$. "enter - oder"
  - "not" with $\neg$. "nicht"
- Thus the above last line can be written as $S(m)$ is true $\forall m \in \mathbb{N}$ or even more briefly as $S(m)\,\forall m \in \mathbb{N}$.

2.1.3 Formal Definition for Induction

- It is an assumption of mathematicians (part of the Peano axioms to be discussed on Wednesday) that:

> IF we prove that:
>
> - $S(1)$ is true
>
>   AND
> - $\forall l \in \mathbb{N}$:
>   - $S(l+1)$ is true whenever (immer wenn) $S(l)$ is true.
>
> THEN: $S(m)$ is true $\forall m \in \mathbb{N}$.

(Ein Paar) Notes

- Clearly this makes sense: If $S(1)$ is true and and $S(l+1)$ is true whenever $S(l)$ is true, then we can see easily that if someone picks out some random $m \in \mathbb{N}$, say, $m = 583$, then we could (if we had time) repeat the recipe 582 times from $S(1)$ up to $S(583)$ and verify that $S(583)$ is true. Because 583 was arbitrary (zufaellig) (the only thing we used was that it was above 1) then we conclude $S(m)$ is true $\forall m \in \mathbb{N}$.
- This shows that we don't have to start at 1 (will be relevant for one of the later questions):

IF we prove that:

- $S(37)$ is true

  AND

- $\forall l \in \{37,\ 38,\ 39,\ 40,\ \dots\}$:

  * $S(l+1)$ is true whenever $S(l)$ is true.

THEN: $S(m)$ is true $\forall m \in \{37,\ 38,\ 39,\ 40,\ \dots\}$.

- Caveat (Achtung): make sure that really given that $S(l)$ is true then $S(l+1)$ is true, FOR ALL $l \in \mathbb{N}$. If the chain breaks at any point then the statement is no longer true.

2.1.4 Example

- Let $S(m)$ be the statement $\sum_{j\in\{1,\ 2,\ \dots,\ m\}} j^3 = \left[\frac{1}{2}m(m+1)\right]^2$.

- First, clarification about notation: $\sum_{j\in\{1,\ 2,\ \dots,\ m\}} j^3 = 1^3 + 2^3 + \cdots + m^3$.

- *Claim 1:* $S(m)$ is true $\forall m \in \mathbb{N}$.
  Proof: (model proof for homework) (Eure Hausaufgabe Loesung sollte so aussehen)

  - Use the induction procedure:
  - *Claim 1.1:* $S(1)$ is true.
    Proof:
    * Compute (Rechnen) the left hand (linke seite) side of the equation when $m = 1$:

    $$\begin{aligned}\sum_{j\in\{1\}} j^3 &= 1^3 \\ &= 1\end{aligned}$$

    * For the right hand side, with $m = 1$, $\left[\frac{1}{2}m(m+1)\right]^2$ is equal to:

    $$\begin{aligned}\left[\frac{1}{2}1(1+1)\right]^2 &= [1]^2 \\ &= 1\end{aligned}$$

    * So we have proven Claim 1.1.
  - *Claim 1.2:* $S(l+1)$ is true whenever $S(l)$, for any $l \in \mathbb{N}$.
    Proof:
    * Let l be given (chosen from $\mathbb{N}$). (l Sei gegeben)
    * Assume that for this given l, $S(l)$ is true.

* Claim 1.2 would be proven if only we could show that $S(l+1)$ is also true (assuming at the same time that $S(l)$ is true).
* To this end, start with the left hand side of the equation for $S(l+1)$:

$$\begin{aligned}\sum_{j\in\{1,\,2,\,\ldots,\,l,\,l+1\}} j^3 &= \underbrace{1^3+2^3+\cdots+l^3}_{\sum_{j\in\{1,\,2,\,\ldots,\,l\}} j^3}+(l+1)^3 \\ &= \sum_{j\in\{1,\,2,\,\ldots,\,l\}} j^3+(l+1)^3\end{aligned}$$

* Because $S(l)$ is true, we know that $\sum_{j\in\{1,\,2,\,\ldots,\,l\}} j^3 = \left[\frac{1}{2}l(l+1)\right]^2$. Plug it in to get:

$$\begin{aligned}\sum_{j\in\{1,\,2,\,\ldots,\,l,\,l+1\}} j^3 &= \underbrace{1^3+2^3+\cdots+l^3}_{\sum_{j\in\{1,\,2,\,\ldots,\,l\}} j^3}+(l+1)^3 \\ &= \sum_{j\in\{1,\,2,\,\ldots,\,l\}} j^3+(l+1)^3 \\ &= \left[\frac{1}{2}l(l+1)\right]^2+(l+1)^3 \\ &= (l+1)^2\left\{\left[\frac{1}{2}l\right]^2+(l+1)\right\} \\ &= (l+1)^2\left\{\frac{1}{4}l^2+l+1\right\} \\ &= \frac{1}{4}(l+1)^2\left\{l^2+4l+4\right\} \\ &= \frac{1}{4}(l+1)^2(l+2)^2 \\ &= \left[\frac{1}{2}(l+1)(l+2)\right]^2\end{aligned}$$

* What we find is that assuming that $S(l)$ is true, it turns out that $S(l+1)$ is also true.
* So we have proven Claim 1.2.

– Because Claim 1.1 and Claim 1.2 have been proven, we may use the induction axiom and conclude that $S(m)$ is true $\forall m \in \mathbb{N}$. ■

2.2 Functions (Funktionen oder Abbildungen)

- Functions are rules (Regeln) sending elements from one set X into another set Y.
- Example:

- Assume: $X = \{Italy, Switzerland, Canada, Mexico\}$ and $Y = \{Europe, America\}$ then we can think of a function that assigns to each country its corresponding continent:

$$\begin{aligned} Italy &\rightarrow Europe \\ Switzerland &\rightarrow Europe \\ Canada &\rightarrow America \\ Mexico &\rightarrow America \end{aligned}$$

- Functions cannot assign two values to one element: not possible that that *Italy* is both in *Europe* and *America*. There are generalized notions of such assignments, but they are not called "function" in the mathematical sense: $X = \{John, David, Alex\}$, $Y = \{Apple, Peach, Banana\}$ and we can think of a function that tells us which fruits does each person like. Of course *John* can like both *Apple*s and *Banana*s. But this is not a mathematical function.

- More concretely for us, we speak of functions from one set of *numbers* to another set of numbers. for example: a function from $X = \mathbb{N}$ to $Y = \mathbb{N}$ given by:

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 3 \\ 3 &\rightarrow 4 \\ &\dots \end{aligned}$$

  Instead of giving this huge table (in this case infinite!) we sometimes write instead "$\boxed{f(n) = n + 1}$ for all $n \in \mathbb{N}$". The thing in the parenthesis is called the argument of the function and the expression (Ausdruck / Begriff) on the right hand side is the *value* of f at n.

2.2.1 Monotonically Increasing Functions (monoton steigende Funktionen)

- Assume we are given a function as f from X to Y, which are some sets of numbers, and we write $f(x) = y$ for all $x \in X$.

- A function is called monotonically increasing if for all $x_1 \in X$ and for all $x_2 \in X$ such that $x_1 \leq x_2$ ("kleiner oder gleich"), we have $f(x_1) \leq f(x_2)$. Thus, f preserves the order.

Example

- *Claim:* The function f from $X = [0, \infty]$ to $Y = [0, \infty]$ given by $f(x) = x^2$ for all $x \in X$ is monotonically increasing.
  Proof:

– Assume we are given two values, $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that $x_1 \leq x_2$. Our goal is to show that $f(x_1) \leq f(x_2)$, that is, that ${x_1}^2 \leq {x_2}^2$.

– Since $x_1 \leq x_2$, we know that $x_2 - x_1 > 0$. So write this as $\alpha = x_2 - x_1$. Thus $x_1 + \alpha = x_2$, and $\alpha \geq 0$. Thus,

$$\begin{aligned} {x_2}^2 &= (x_1 + \alpha)^2 \\ &= {x_1}^2 + 2\alpha x_1 + \alpha^2 \end{aligned}$$

  * Now observe that α^2 is either positive or zero, α is positive or zero, and x_1 is positive. So that ${x_2}^2 \geq {x_1}^2$.

$\blacksquare$

2.3 Set Theoretic Operations (Menge Theorie Operationen)

So far we have introduced some sporadic sets: $\mathbb{N}$, $\mathbb{R}$ (the real numbers), $\{John, David, Alex\}$. There are ways to obtain new sets from given sets:

- Set difference: $X \backslash Y$ denotes the set containing all the elements in X which are NOT in Y. For example: $X = \mathbb{N}$ and $Y = \{2, 4, 6, \ldots\}$. Then $X \backslash Y = \{1, 3, 5, \ldots\}$.

- Union: $X \cup Y$ denotes the set containing all the elements EITHER in X OR in Y. For example: $X = \{John, David\}$ and $Y = \{Alex, Manfred\}$ then $X \cup Y = \{John, David, Alex, Manfred\}$.

- Intersection: $X \cap Y$ denotes all the elements BOTH in X AND in Y: For example: $X = \{John, David, Manfred\}$ and $Y = \{Alex, Manfred\}$ then $X \cap Y = \{Manfred\}$.

- Empty set: the result of each of these operations can lead to a situation where there are NO elements fitting the description. For example, what happens if $X = \{John, David\}$ and $Y = \{Alex, Manfred\}$ and we are interested in $X \cap Y$? According to the above, there are NO such elements and so we must admit that $X \cap Y$ is "empty", and is really called "the empty set" and denoted by $\varnothing$. So $X \cap Y = \varnothing$.

- Complement: If we have many different sets: X, Y_1, Y_2, Y_3 and so on, there is a shortcut to write $X \backslash Y_1$, $X \backslash Y_2$, $X \backslash Y_3$ and so on. Instead, we write ${Y_1}^c$, ${Y_2}^c$, ${Y_3}^c$. In this notation X is hidden, but must be remembered.

- Subsets: Sometimes when we have two sets X and Y, all the elements of Y can be also elements in X. We say then that Y is a subset of X and write $X \subset Y$. This notation does not exclude that $X = Y$, that is, that all the elements of X are also elements of Y.

Equality of Sets

- There is an axiom (maybe more on Wednesday) saying that a set is completely determined by its elements. Thus, to prove that two sets are equal we must show that they have the same list of elements. This is sometimes hard, so instead we prove a logically equivalent statement:
  - X and Y are the same set if and only if (wenn, und nur dann wenn) all the elements in X are also elements in Y and all the elements in Y are also elements in X. In symbols: $\boxed{X = Y \iff ((X \subset Y) \wedge (Y \subset X))}$.
  - Thus to prove $X = Y$ we must proceed in two steps:
    - ∗ Step 1: $X \subset Y$:
      - · Every element of X should be an element of Y also.
      - · So pick some element of X, at random, $x \in X$, and show that $x \in Y$ holds as well. Because this element was chosen arbitrarily, the statement holds true of ALL elements.
    - ∗ Step 2: $Y \subset X$: (same logic in reverse)
      - · Every element of Y should be an element of X also.
      - · So pick some element of Y, at random, $y \in Y$, and show that $y \in X$ holds as well. Because this element was chosen arbitrarily, the statement holds true of ALL elements.
  - Note: X and Y can be composite objects composed with the set theoretic operations. For example, we might have to prove something like $(A \cap B) \cap C = A \cap (B \cap C)$. To prove this we proceed exactly in the same way, where $X = (A \cap B) \cap C$ and $Y = A \cap (B \cap C)$.
  - *Claim:* $(A \cap B) \cap C = A \cap (B \cap C)$
    Proof:
    - ∗ Step 1: $(A \cap B) \cap C \subset A \cap (B \cap C)$
      - · Take $x \in (A \cap B) \cap C$ as some random element.
      - · That means that $x \in A \cap B$ and $x \in C$.
      - · That means that ($x \in A$ and $x \in B$) and $x \in C$.
      - · But the order of the logical statements about or doesn't matter with parenthesis, so we could just as well write: that $x \in A$ and ($x \in B$ and $x \in C$).
      - · That means that $x \in A$ and $x \in B \cap C$.
      - · That means that $x \in A \cap (B \cap C)$.
    - ∗ Step 2: works in the same way.

Example

- This example is useful for the very last question.

- If we have a set of sets, denote the big set (of sets) as $\mathcal{B}$ and each set belonging to $\mathcal{B}$ as $X_1, X_2, X_3, \ldots$. That is, $\mathcal{B} = \{X_1, X_2, X_3, \ldots\}$, and, in addition, each X_i itself contains elements: $X_1 = \{a, b\}$, $X_2 = \{\alpha, \beta, \gamma\}$ and so on.
- What if we wanted to count the total number of elements of all the constituent sets in $\mathcal{B}$, that is, the number of elements in X_1 is 2, the number of elements in $X_2 = 3$. So far we have 5. What if we wanted to add all of this up? $|X_1| + |X_2| + \ldots$. The absolute value sign on sets denotes the number of elements.
- Important assumption: No two sets in $\mathcal{B}$ have the same number of elements (Gleiche Anzahl von Elementen)!
- In addition, we know that $\mathcal{B}$ has $|\mathcal{B}|$ sets in it (by definition).
- Thus, the number we are looking for is $\sum_{i \in \{1,\, 2,\, \ldots,\, |\mathcal{B}|\}} |X_i|$.
- Since no two sets have the same size, the very worst (smallest) case is if X_1 is empty, X_2 has one element, X_3 has two elements, and so on. Thus, $\sum_{i \in \{1,\, 2,\, \ldots,\, |\mathcal{B}|\}} |X_i| \geq \sum_{k \in \{0,\, \ldots,\, |\mathcal{B}|-1\}} k = \frac{1}{2} |\mathcal{B}| \left(|\mathcal{B}| - 1\right)$.

2.4 The Integers

The integers, denoted by $\mathbb{Z}$, are built from the natural numbers $\mathbb{N}$ by going in the negative direction and including 0: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

2.5 The Rationals

The rational numbers, denoted by $\mathbb{Q}$, are built from all possible ratios of two integers, where the denominator is not zero, and reducing any possible mutual factors factors. So all numbers of the form $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$ and where a and b have no mutual factors. (So $\frac{2}{4}$ will be reduced to $\frac{1}{2}$).

- Every decimal number that terminates is representible as the ratio of two integers (just multiply by a big enough power of 10 and divide:

$$
\begin{aligned}
0.125 &= \frac{0.125 \times 10^3}{10^3} \\
&= \frac{125}{1000} \\
&= \frac{5^3}{8\,5^3} \\
&= \frac{1}{8}
\end{aligned}
$$

- Every decimal number that is periodic is also representible as such a ratio. $0.333333\cdots = \frac{1}{3}$.

- Decimal numbers that do not terminate and are not periodic are not rational! $e = 2.71828 \dots = ??$ not possible.
- The rationals form a field: closed under addition and multiplication (among other properties of fields). So if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $(x + y) \in \mathbb{Q}$.

Example

- *Claim:* $\sqrt{2}$ is not rational.
  Proof:
  - Assume that $\sqrt{2}$ *is* rational, that is $\sqrt{2} \in \mathbb{Q}$.
  - Then we can find some $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$ such that a and b have no mutual factors (gegenseitig Faktoren) and $\sqrt{2} = \frac{a}{b}$.
  - Compute

$$\begin{aligned} \left(\sqrt{2}\right)^2 &= \left(\frac{a}{b}\right)^2 \\ 2 &= \frac{a^2}{b^2} \\ 2b^2 &= a^2 \end{aligned}$$

  - So that a^2 is even (gerade).
  - So that a is even (otherwise $a = 2m + 1$ for some m, and so $a^2 = 4m^2 + 4m + 1$ which is *not* even).
  - So that a^2 has a factor of 4.
  - So that $2b^2$ has a factor of 4, because we found $2b^2 = a^2$.
  - Thus b^2 is even. Thus b is even.
  - So both a and b are even, and so they have a mutual factor (2) and so we reached a contradiction. So that the original assumption (that $\sqrt{2}$ is rational) must be false.

  ■

2.6 The Reals

The real numbers, denoted by $\mathbb{R}$, contain everything else (and including the rationals). It is not simple to prove that this exists and that it is unique (perhaps we will do it). So, $\pi \in \mathbb{R} \setminus \mathbb{Q}$ and $e \in \mathbb{R} \setminus \mathbb{Q}$ for example.

2.7 Absolute Value (Betragsfunktion)

The absolute value of a number $|x|$, is defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. Whenever you see an inequality, split it into two cases.

2.7.1 Examples

- $|5| = 5$ because $5 > 0$.
- $|-3| = -(-3) = 3$ because $-3 < 0$.

2.7.2 Use with Inequalities

If we have an equation $|x| \leq y$, we can split it into two cases:

- If $x \geq 0$, we have: $\boxed{x \leq y}$
- if $x < 0$ then we have $-x \leq y$ which means $\boxed{x \geq -y}$.

Thus we can write these two cases together as $\boxed{-y \leq x \leq y}$.

- Use this notation whenever you have an inequality with absolute values. Same procedure for $|x| \geq y$.

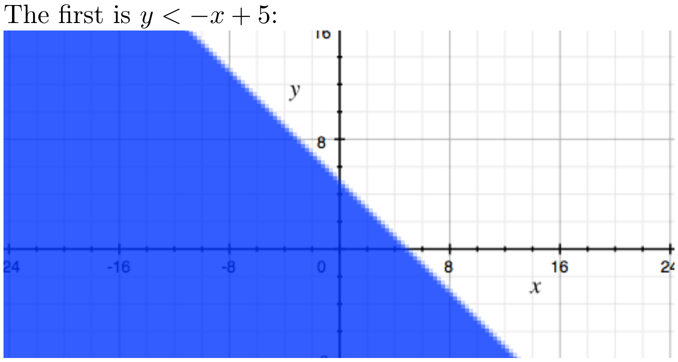
Example

- Solve $\left|x^2 + 2\right| \leq 6$.
- Write instead as $-6 \leq x^2 + 2 \leq 6$ which is the same as $\begin{cases} x^2 - 4 \leq 0 \\ x^2 + 8 \geq 0 \end{cases}$.
- The first equation is true between $[-2, 2]$ and the second is *always* true, so our solution is $-2 \leq x \leq 2$.

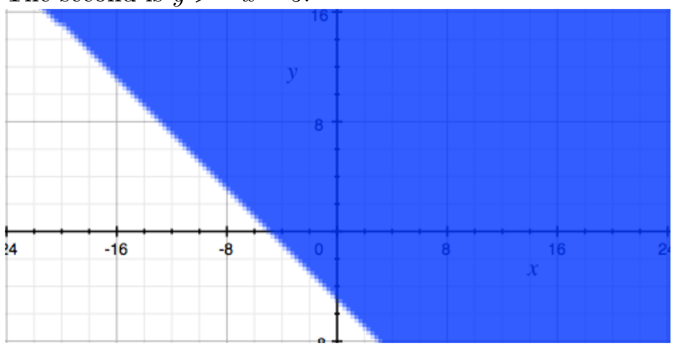
2.7.3 Example with Sets

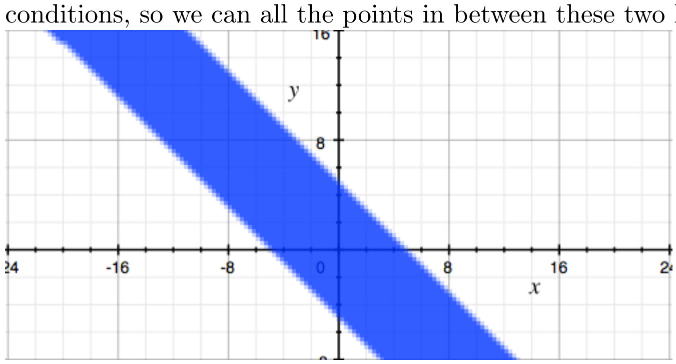
What does the set $\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; |x + y| < 5 \right\}$ describe?

- Use again $-5 < x + y < 5$.
- So two statements have to hold for all points: $x + y < 5$ AND $x + y > -5$.
- That is,
- The first is $y < -x + 5$:



AND

- The second is $y > -x - 5$:


- This describes two lines and we want all the points that obey the two conditions, so we can all the points in between these two lines:


2.8 The Triangle Inequality (Dreiecksungleichung)

Given two real numbers $x \in \mathbb{R}$ and $y \in \mathbb{R}$, there is a very useful relation (which you will use many many times):

- *Claim:* $\boxed{|x+y| \leq |x| + |y|}$.
  Proof:
  - It is generally true that $\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases}$.
  - Add these two statements together to get $-|x| - |y| \leq x + y \leq |x| + |y|$ which is equivalent to $-(|x| + |y|) \leq x + y \leq |x| + |y|$.
  - But we have just seen that $|\alpha| \leq \beta$ is equivalent to $-\beta \leq \alpha \leq \beta$ for any two real numbers α and β.
  - Thus we conclude that $|x + y| \leq |x| + y$.

2.9 Omitting (Auslassen) the First Zero Summand in a Sum

If we have a sum, given by $\sum_{k\in\{0,\,1,\,2,\,3,\,\ldots,\,n\}} f(k) = f(0) + f(1) + f(2) + \cdots + f(n)$, then of course, if $f(0) = 0$ we can exclude it from the sum, and so we have: $\sum_{k\in\{1,\,2,\,3,\,\ldots,\,n\}} f(k) = f(1) + f(2) + \cdots + f(n)$.

Example

- If $f(k) = k$, then of course $0 + 1 + 2 + 3 + \cdots + n = 1 + 2 + 3 + \cdots + n$.

2.10 Change of Variables in Sums (Substitution in Summen)

Given a sum $\sum_{k\in\{0,\,1,\,2,\,3,\,\ldots,\,n\}} f(k) = f(0) + f(1) + f(2) + \cdots + f(n)$, we can shift the variable of summation k:

- We could also write:

$$\begin{aligned}\sum_{k\in\{0,\,1,\,2,\,3,\,\ldots,\,n\}} f(k) &= f(0) + f(1) + f(2) + \cdots + f(n) \\ &= \sum_{k\in\{-1,\,0,\,1,\,2,\,\ldots,\,n-1\}} f(k+1) \\ &= f(0) + f(1) + f(2) + \cdots + f(n)\end{aligned}$$

- See how we never actually have to compute $f(-1)$ or $f(n+1)$ (which may be undefined) because the shifting of the argument $k \mapsto k+1$ fixes the shift.
- Conclusion: $\sum_{k\in\{0,\,1,\,2,\,3,\,\ldots,\,n\}} f(k) = \sum_{k\in\{-1,\,0,\,1,\,2,\,\ldots,\,n-1\}} f(k+1)$.
- Can also shift the other direction.

2.11 Partial Fraction Decomposition (Partialbruchzerlegung)

- Sometimes we want to take a complicated fraction (Bruchrechnung) and decompose it: $\frac{7k-13}{(k+3)(k+2)} = \frac{8(k+2)-(k+3)}{(k+3)(k+2)} = \frac{8}{k+3} - \frac{1}{k+2}$. (This is useful for one of the later questions).
- How to do this?
- Assume that we *can* do this and make an ansatz: $\frac{a}{p(k)q(k)} = \frac{X}{p(k)} + \frac{Y}{q(k)}$. But what are X and Y?
- Try to solve: $\frac{q(k)Y+p(k)X}{p(k)q(k)}$. Compare $q(k)\,Y + p(k)\,X = a$ coefficients of k to find two equations for X and Y.

Example

- Want to decompose $\frac{k}{(k+5)(k-8)}$.
- Assume we can write: $\frac{k}{(k+5)(k-8)} = \frac{X}{k+5} + \frac{Y}{k-8}$.
- Then $\frac{X}{k+5} + \frac{Y}{k-8} = \frac{X(k-8)+Y(k+5)}{(k+5)(k-8)} = \frac{(X+Y)k-8X+5Y}{(k+5)(k-8)} \stackrel{!}{=} \frac{k}{(k+5)(k-8)}$
- So that $X+Y = 1$ and so $Y = 1-X$. The second equation is $-8X+5Y = 0$ and so $-8X + 5\,(1-X) = 0$ or $X = \frac{5}{13}$ and $Y = \frac{8X}{5} = \frac{8}{13}$.
- Check that it works: $\frac{\frac{5}{13}}{k+5} + \frac{\frac{8}{13}}{k-8} = \frac{\frac{5}{13}(k-8)+\frac{8}{13}(k+5)}{(k+5)(k-8)} = \frac{k}{(k+5)(k-8)}$.