Analysis 1
Recitation Session of Week 10

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1 Exercise Sheet Number 8

1.1 Question 1

- Let $s \in \mathbb{Q}$ be given.
  Claim: The map $f:(0,\infty)\to\mathbb{R}$ given by $x\mapsto x^s$ is continuous.
  Proof:
  - Note: You *may not* use the fact that if f and g are continuous then so is their multiplication map, because $s\in\mathbb{Q}$ and not necessarily in $\mathbb{Z}$, so you may not write $x^s = \underbrace{x\cdot x\cdot x\cdot\dots\cdot x}_{s\text{-times}}$.
  - So we know the map is continuous for $s\in\mathbb{Z}$ so assume $s\notin\mathbb{Z}$ and write $s=\frac{p}{q}$ where $\gcd(p,\,q)=1$, $p\in\mathbb{Z}$ and $q\in\mathbb{N}\setminus\{0\}$.
  - We can write $x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p$, and again, we know that $x\mapsto x^p$ is continuous when $p\in\mathbb{Z}$, so WLOG we may assume that $p=1$ (using the fact that composition of continuous functions is continuous).
  - Thus our goal is reduced to prove that $x\mapsto x^{\frac{1}{q}}$ where $q\in\mathbb{N}\setminus\{0\}$ is continuous at x for all $x\neq 0$.
  - So let $\varepsilon>0$ be given and let some $x_0\in(0,\infty)$ be given.
  - Take $\delta(x_0,\,\varepsilon) := \varepsilon\left|{x_0}^{\frac{1}{q}-1}\right|$.
  - Then if $|x-x_0| < \varepsilon\left|{x_0}^{\frac{1}{q}-1}\right|$, we have

$$\begin{aligned}\left|x^{\frac{1}{q}}-{x_0}^{\frac{1}{q}}\right| &= \left|\frac{x-x_0}{x^{\frac{1}{q}-1}+x^{\frac{1}{q}-2}x_0+\dots+x{x_0}^{\frac{1}{q}-2}+{x_0}^{\frac{1}{q}-1}}\right| \\ &\leq \frac{\varepsilon\left|{x_0}^{\frac{1}{q}-1}\right|}{\left|x^{\frac{1}{q}-1}+x^{\frac{1}{q}-2}x_0+\dots+x{x_0}^{\frac{1}{q}-2}+{x_0}^{\frac{1}{q}-1}\right|} \\ &\leq \frac{\varepsilon\left|{x_0}^{\frac{1}{q}-1}\right|}{\left|{x_0}^{\frac{1}{q}-1}\right|} \\ &\leq \varepsilon_0\end{aligned}$$

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- Part (b): *Claim*: $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ has continuous extension on the whole of $\mathbb{C}$ when $s<1$.
  Proof:
  - In order to have an analytic extension, we need this new function $F:\mathbb{C}\to\mathbb{C}$ to obey the following two conditions:
    1. F has to be continuous on the whole of $\mathbb{C}$.
    2. F has to agree with f for the domain of f, $\mathbb{C}\setminus\{0\}$.

  1. Thus define $F:\mathbb{C}\to\mathbb{C}$ as $z\mapsto\begin{cases} f(z) & z\in\mathbb{C}\setminus\{0\} \\ w & z=0\end{cases}$.
  2. The only question that remains is what should this $w\in\mathbb{C}$ be, and the way to find out, is to demand that F is continuous at 0.
  3. For functions $\mathbb{C}\to\mathbb{C}$, continuity is equivalent to sequential continuity, so that we may just as well demand that $\lim_{z\to 0}F(z)\overset{!}{=}w$.
  4. But $\lim_{z\to 0}F(z)=\lim_{z\to 0}f(z)$ because F and f agree for all $z\neq 0$.

5. Thus we need to compute $\lim_{z\to0} f(z)$.
6. If this limit exists then it should not depend on how we approach zero (theorem 4.2 in Rudin). In particular, we may approach zero via the real axis:

$$\begin{aligned}\lim_{z\to0} f(z) &= \lim_{R\to0}\frac{\overline{R}}{|R|^s}\\ &= \lim_{R\to0} R^{1-s}\\ &\overset{t\mapsto t^{1-s}\text{ is continuous}}{=} \left(\lim_{R\to0}R\right)^{1-s}\\ &= 0^{1-s}\\ &= 0\end{aligned}$$

where $R\in(0,\infty)$
7. Hence the limit exists, and thus if we define $w=0$ then F is indeed continuous at 0 and we are set.

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- This couldn't have worked for $s\geq1$ because then the limit $\lim_{z\to0} f(z)$ either diverges or does not exist.

1.2 Question 2

- *Claim*: $f:\mathbb{C}\backslash\mathbb{Z}\to\mathbb{C}$ defined by $z\mapsto\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$ is continuous and $f(z)=f(z+1)$.
Note: There is an identity saying that $\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$ but you are not supposed to know that.
Proof:
  - Define the partial sums $f_N(z):=\frac{1}{z}+\sum_{n=1}^{N}\frac{2z}{z^2-n^2}$ for all $N\in\mathbb{N}$.
  - Define

$$\begin{aligned}M_N &:= \sup\left(\{\,|f_N(z)-f(z)|\mid z\in\mathbb{C}\backslash\mathbb{Z}\,\}\right)\\ &= \sup\left(\left\{\left|\frac{1}{z}+\sum_{n=1}^{N}\frac{2z}{z^2-n^2}-\pi\cot(\pi z)\right|\ \middle|\ z\in\mathbb{C}\backslash\mathbb{Z}\right\}\right)\end{aligned}$$

  - We know that $f_N\to f$ uniformly on $\mathbb{C}\backslash\mathbb{Z}$ if and only if $M_N\to0$ as $N\to\infty$ (theorem 7.9 in Rudin).
  - But $M_N=\infty$ clearly, so that it does not converge to zero!
  - Thus f_N cannot converge uniformly to f, and we may not use uniform convergence to conclude continuity of f.
  - Instead, what you should have done is tried to prove uniform continuity on some subset of $\mathbb{C}\backslash\mathbb{Z}$.
  - Let $z\in\mathbb{C}\backslash\mathbb{Z}$ be given, and pick some $\varepsilon>0$ so that $\overline{B_\varepsilon(z)}\equiv\{\,\omega\in\mathbb{C}\mid|z-w|\leq\varepsilon\,\}\subseteq\mathbb{C}\backslash\mathbb{Z}$.
    - This is possible because $(\mathbb{C}\backslash\mathbb{Z})\in\mathit{Open}(\mathbb{C})$ (because $\mathbb{Z}\in\mathit{Closed}(\mathbb{C})$ (because a singleton $\{\,z_0\,\}\in\mathit{Closed}(\mathbb{C})$ for all $z_0\in\mathbb{C}$ and $\mathbb{Z}$ is a union of closed such singletons)).
  - *Claim*: $f_N|_{\overline{B_\varepsilon(z)}}\to f|_{\overline{B_\varepsilon(z)}}$ uniformly.
  Proof:
    - Choose $N_1\in\mathbb{N}$ so that $2(|z|+\varepsilon)\leq N_1$. Then for all $N>N_1$ we have

$$\begin{aligned}\tilde{M}_N &:= \sup\left(\left\{\left|f_N|_{\overline{B_\varepsilon(z)}}(w)-f|_{\overline{B_\varepsilon(z)}}(w)\right|\ \middle|\ w\in\overline{B_\varepsilon(z)}\right\}\right)\\ &= \sup\left(\left\{\left|\sum_{n=N+1}^{\infty}\frac{2w}{w^2-n^2}\right|\ \middle|\ w\in\overline{B_\varepsilon(z)}\right\}\right)\\ &\leq \sup\left(\left\{\sum_{n=N+1}^{\infty}\left|\frac{2w}{w^2-n^2}\right|\ \middle|\ w\in\overline{B_\varepsilon(z)}\right\}\right)\\ &= \sup\left(\left\{2|w|\sum_{n=N+1}^{\infty}\frac{1}{n^2}\left|\frac{1}{\frac{w^2}{n^2}-1}\right|\ \middle|\ w\in\overline{B_\varepsilon(z)}\right\}\right)\\ &\leq \sup\left(\left\{2(|z|+\varepsilon)\sum_{n=N+1}^{\infty}\frac{1}{n^2}\left|\frac{1}{\frac{(|z|+\varepsilon)^2}{{N_1}^2}-1}\right|\ \middle|\ w\in\overline{B_\varepsilon(z)}\right\}\right)\\ &= \sup\left(\left\{\underbrace{2(|z|+\varepsilon)\left|\frac{1}{\frac{(|z|+\varepsilon)^2}{{N_1}^2}-1}\right|}_{\text{bounded}}\underbrace{\sum_{n=N+1}^{\infty}\frac{1}{n^2}}_{\text{converges to zero as }N\to\infty}\ \middle|\ w\in\overline{B_\varepsilon(z)}\right\}\right)\\ &\to 0\end{aligned}$$

- Thus we can conclude that $f|_{\overline{B_\varepsilon(z)}}$ is continuous because $f_N|_{\overline{B_\varepsilon(z)}}$ are all continuous.
- *Claim*: If $f|_{\overline{B_\varepsilon(z)}}$ is continuous at z then f is continuous at z. (homework).
- But z was arbitrary, so that f is continuous for all $z \in \mathbb{C}\backslash\mathbb{Z}$.

1.3 Question 3

- Let A be some countable subset of $\mathbb{R}$, and let $\sum_{n=1}^{\infty} s_n$ be an absolutely convergent series of real numbers.
- Define $f(x) := \sum_{n=1}^{\infty} s_n sign(x-a_n)$ where

$$sign(x) \equiv \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

- *Claim*: The partial sums $f_N \equiv \sum_{n=1}^{N} s_n sign(x-a_n)$ converge uniformly to f.
  Proof:
  - Use the Weierstrass M test with $M_n \equiv s_n$.
- *Claim*: f is continuous on $\mathbb{R}\backslash A$.
  Proof:
  - Follows from uniform convergence.
- *Claim*: $[\lim_{\varepsilon\to 0} f(a_n+\varepsilon)] - [\lim_{\varepsilon\to 0} f(a_n-\varepsilon)] = 2s_n$.
  Proof:
  - Make the calculation

$$\begin{aligned} \lim_{\varepsilon\to 0} f(a_n+\varepsilon) &= \lim_{\varepsilon\to 0}\lim_{N\to\infty} f_N(a_n+\varepsilon) \\ &= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\sum_{j=1}^{N} s_j sign(a_n+\varepsilon-a_j) \\ &= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\left\{ s_n \underbrace{sign(\varepsilon)}_{1} + \sum_{j=1, j\neq n}^{N} s_j sign(a_n+\varepsilon-a_j)\right\} \\ &= s_n + \underbrace{\lim_{\varepsilon\to 0}\lim_{N\to\infty}\sum_{j=1, j\neq n}^{N} s_j sign(a_n+\varepsilon-a_j)}_{P} \end{aligned}$$

  - In a very similar fashion we can calculate that $\lim_{\varepsilon\to 0} f(a_n-\varepsilon) = -s_n+P$.
  - Still need to show that P exists to make this reigorous. Have a look in the official solutions for details.
- *Claim*: If $s_n>0$ for all $n\in\mathbb{N}$ then f is monotonically increasing.
  Proof:
  - The function $x\mapsto s_n sign(x-a_n)$ is monotonically increasing for any n (homework).
  - The sum of monotone increasing functions is monotone increasing.
  - Due to $a_n \le b_n \implies \lim a_n \le \lim b_n$ we have that f is monotonically increasing.

1.4 Question 4

- Almost everyone did it well. Just remember that you must define the domain of a function whenever you are defining a function.

1.5 Question 5

- Let X and Y be metric spaces, and let $(A_j)_{j=0}^{n-1} \subseteq Closed(X)$ for some $n\in\mathbb{N}$. Define $A := \bigcup_{j=0}^{n-1} A_j$.
- Part (a): *Claim*: $f: A\to Y$ is continuous if and only if $f|_{A_i}: A_i \to Y$ is continuous for all $i\in\mathbb{Z}_n$.
  Proof:
  - $\boxed{\implies}$
    * Let $i\in\mathbb{Z}_n$.

    - $*$ We know that $f: A \to Y$ is continuous. Thus, $\forall x \in A, \forall \varepsilon > 0\ \exists \delta_f(\varepsilon, x) > 0$ such that if $\tilde{x} \in B_{\delta_f(\varepsilon, x)}(x)$ then $f(\tilde{x}) \in B_\varepsilon(f(x))$.
    - $*$ Let $\varepsilon > 0$ be given, and let $x \in A_i$ be given.
    - $*$ Take $\delta_{f|_{A_i}}(x, \varepsilon) := \delta_f(x, \epsilon)$.
    - $*$ Then if $\tilde{x} \in B_{\delta_{f|_{A_i}}(x, \varepsilon)}(x) \cap A_i$ then $f(\tilde{x}) \in B_\varepsilon(f(x))$ which implies $f|_{A_i}(\tilde{x}) \in B_\varepsilon\left(f|_{A_i}(x)\right)$ because both x and $\tilde{x}$ lie in A_i.
  - $\boxed{\Longleftarrow}$
    - $*$ Let $x \in A$ and some $\varepsilon > 0$ be given.
    - $*$ Define $I := \{\, i \in \mathbb{Z}_n \mid x \in A_i \,\}$.
    - $*$ $f|_{A_i}$ is continuous at x for all $i \in I$.
    - $*$ Then if $\tilde{x} \in B_{\delta_{f|_{A_i}}(x, \varepsilon)}(x) \cap A_i$ then $f|_{A_i}(\tilde{x}) \in B_\varepsilon\left(f|_{A_i}(x)\right)$ for all $i \in I$ (there exist such $\delta_{f|_{A_i}}(x, \varepsilon)$).
    - $*$ From this it follows that if $\tilde{x} \in B_{\delta_{f|_{A_i}}(x, \varepsilon)}(x) \cap A_i$ then $f(\tilde{x}) \in B_\varepsilon(f(x))$ for all $i \in I$ (there exist such $\delta_{f|_{A_i}}(x, \varepsilon)$).
    - $*$ Define $\tilde{\delta}(x, \varepsilon) := \min\left(\left\{\, \delta_{f|_{A_i}}(x, \varepsilon) \,\middle|\, i \in I \,\right\}\right)$.
    - $*$ Then if $\tilde{x} \in B_{\tilde{\delta}(x, \varepsilon)}(x) \cap \left(\bigcup_{i \in I} A_i\right)$ then $f(\tilde{x}) \in B_\varepsilon(f(x))$.
    - $*$ Define $J := \mathbb{Z}_n \backslash I$.
    - $*$ Define $C := \bigcup_{i \in J} A_i$.
    - $*$ *Claim*: $C \in Closed(X)$.
      Proof:
      - $\cdot$ C is a *finite* union of closed subsets of X. The property of being closed is "closed" under finite unions.
    - $*$ *Claim*: $x \notin C$.
      Proof:
      - $\cdot$ By definition of I.
    - $*$ Thus $(X \backslash C) \in Open(X)$ such that $x \in (X \backslash C)$.
    - $*$ Thus, $\exists \tilde{\tilde{\delta}}(x, \varepsilon) > 0$ such that $B_{\tilde{\tilde{\delta}}(x, \varepsilon)}(x) \subseteq (X \backslash C)$.
    - $*$ Thus, $B_{\tilde{\tilde{\delta}}(x, \varepsilon)}(x) \cap C = \varnothing$.
    - $*$ Define $\delta(x, \varepsilon) := \min\left(\left\{\tilde{\delta}(x, \varepsilon), \tilde{\tilde{\delta}}(x, \varepsilon)\right\}\right)$.
    - $*$ Thus if $\tilde{x} \in B_{\delta(x, \varepsilon)}(x)$, then $\tilde{x} \notin C$ and so $\tilde{x} \in \left(\bigcup_{i \in I} A_i\right)$, *and* $x \in B_{\tilde{\delta}(x, \varepsilon)}(x)$ so that we may conclude $f(\tilde{x}) \in B_\varepsilon(f(x))$.

- For part (b):
  - Define $A_0 = [0, \infty)$ and $A_1 = (-\infty, 0)$, and define $f|_{A_0} := (x \mapsto 1)$ and $f|_{A_1} := (x \mapsto 0)$. Then define $f: \mathbb{R} \to \mathbb{R}$ as in (a), where $A_0 \cup A_1 = \mathbb{R}$.
  - Because the restrictions $f|_{A_0}$ and $f|_{A_1}$ are constant they are continuous, yet, f is not continuous at 0.4

1.6 Question 6

- This was largely covered in the colloquium on the Cantor set. You may read the summary of that colloquium and also the official solutions to the exercises.

2 Exercise Sheet Number 10

2.1 Differentiation

Let $f: [a, b] \to \mathbb{R}$ be a function. Then for any $x \in [a, b]$ define

$$\boxed{f'(x) \equiv \lim_{t \to x} \frac{f(t) - f(x)}{t - x}}$$

if the limit exists.

- If the limit exists, we say that f is differentiable at x, and that f' is its derivative at x.
- *Claim*: If f is differentiable at $x \in [a, b]$ then f is continuous at x.
  Proof:

– Use the limit characterization of continuity:

$$\begin{aligned}\lim_{t\to x} f(t) &= \lim_{t\to x}\left[f(t)-f(x)+f(x)\right]\\ &= \lim_{t\to x}\left[\frac{f(t)-f(x)}{t-x}(t-x)+f(x)\right]\\ &= \lim_{t\to x}\left[\frac{f(t)-f(x)}{t-x}(t-x)\right]+f(x)\\ &= \left[\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\right]\left[\lim_{t\to x}(t-x)\right]+f(x)\\ &= f'(x)\cdot 0+f(x)\\ &= f(x)\end{aligned}$$

- The converse of this theorem is false! (Think about $x\mapsto |x|$ at 0).
- Example: Define $f:\mathbb{R}\to\mathbb{R}$ by $x\mapsto x^2$. Then

$$\begin{aligned}f'(x) &= \lim_{t\to x}\frac{t^2-x^2}{t-x}\\ &= \lim_{t\to x}(t+x)\\ &= 2x\end{aligned}$$

2.2 Concrete Tips for the Homework Exercises

2.2.1 Question 1

- for part (a) use the bionomial formula on $[\cos(x)]^n=\left[\frac{e^{ix}+e^{-ix}}{2}\right]^n$
- For part (b) use
  1. induction
  2. the identity $\cos((n+1)x)=2\cos(x)\cos(nx)+\cos((n-1)x)$ (which you can verify easily).

2.2.2 Question 2

- Calculate $\lim_{x\to\pm\frac{\pi}{2}}\tan(x)$ (from above or from below, depending on whether the plus or minus signs are chosen).
- Use the intermediate value theorem.

2.2.3 Question 3

- Use induction together with:
  1. the "ordinary" Leibniz rool.
  2. the fact that $\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$

2.2.4 Question 4

- May not use the intermediate value theorem, because f' is not necessarily continuous!

2.2.5 Question 5

- Define $\forall k\in\mathbb{Z}_n\equiv\{0,\ldots,n-1\}$ $f_k(x):=\left[\left(1-x^2\right)^n\right]^{(k)}$.
- Then $P_n(x)=\frac{1}{2^n n!}f_n(x)$.
- Show that $f_k(-1)=0=f_k(1)$.
- For part (b):
  - Define $f(x):=\left(x^2-1\right)p'(x)$ where $p(x):=\left(x^2-1\right)^n$.
  - Compute $f^{(n+1)}(x)$ once with $f(x)=\left(x^2-1\right)p'(x)$ and once with $f(x)=2nxp(x)$, and substract the two equations you get.
  - Mulitply by ... to get the desired equation.
  - Use question 3 (a).

2.2.6 Question 6

- Compute $\lim_{x\to\pm\infty} f(x)$.
- Show that $f'(x) > 0$ for all x.
- Compute $f''(x)$ and conclude where f is concave and where it is convex.

2.2.7 Question 7

- For $t = 0$ you must compute the derivative by the actual definition.
- Show f' is not continuous at 0.
- Define $t_k := \frac{1}{(2k+1)\pi}$ and show that $\lim_{k\to\infty} t_k = 0$ and $f'(t_k) = 3$ for all $k \in \mathbb{N}$.