1 Exercise Sheet Number 8

1.1 Question 1

- Let $s \in \mathbb{Q}$ be given.

**Claim:** The map $f : (0, \infty) \to \mathbb{R}$ given by $x \mapsto x^s$ is continuous.

**Proof:**

- Note: You *may not* use the fact that if $f$ and $g$ are continuous then so is their multiplication map, because $s \in \mathbb{Q}$ and not necessarily in $\mathbb{Z}$, so you may not write $x^s = x \cdot x \cdot \cdots \cdot x$ $s$-times.

- So we know the map is continuous for $s \in \mathbb{Z}$ so assume $s \not\in \mathbb{Z}$ and write $s = \frac{p}{q}$ where $\gcd(p, q) = 1, p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$.

- We can write $x^\frac{p}{q} = \left(x^\frac{1}{q}\right)^p$, and again, we know that $x \mapsto x^p$ is continuous when $p \in \mathbb{Z}$, so WLOG we may assume that $p = 1$ (using the fact that composition of continuous functions is continuous).

- Thus our goal is reduced to prove that $x \mapsto x^\frac{1}{q}$ where $q \in \mathbb{N} \setminus \{0\}$ is continuous at $x$ for all $x \neq 0$.

- So let $\varepsilon > 0$ be given and let some $x_0 \in (0, \infty)$ be given.

- Take $\delta(x_0, \varepsilon) := \varepsilon \left| x_0^{\frac{1}{q}-1} \right|$.

- Then if $|x - x_0| < \varepsilon \left| x_0^{\frac{1}{q}-1} \right|$, we have

$$|x^\frac{1}{q} - x_0^\frac{1}{q}| = \left| \frac{x - x_0}{x^{\frac{1}{q}-1} + x^{\frac{1}{q}-2}x_0 + \cdots + x_0^{\frac{1}{q}-2} + x_0^{\frac{1}{q}-1}} \right| \leq \varepsilon \left| x_0^{\frac{1}{q}-1} \right| \leq \varepsilon.$$

- Part (b): **Claim:** $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ has continuous extension on the whole of $\mathbb{C}$ when $s < 1$.

**Proof:**

- In order to have an analytic extension, we need this new function $F : \mathbb{C} \to \mathbb{C}$ to obey the following two conditions:

  1. $F$ has to be continuous on the whole of $\mathbb{C}$.
  2. $F$ has to agree with $f$ for the domain of $f$, $\mathbb{C} \setminus \{0\}$.

  1. Thus define $F : \mathbb{C} \to \mathbb{C}$ as $z \mapsto \begin{cases} f(z) & z \in \mathbb{C} \setminus \{0\} \\ w & z = 0 \end{cases}$.

  2. The only question that remains is what should this $w \in \mathbb{C}$ be, and the way to find out, is to demand that $F$ is continuous at $0$.

  3. For functions $\mathbb{C} \to \mathbb{C}$, continuity is equivalent to sequential continuity, so that we may just as well demand that $\lim_{z \to 0} F(z) = \lim_{z \to 0} f(z)$.

  4. But $\lim_{z \to 0} F(z) = \lim_{z \to 0} f(z)$ because $F$ and $f$ agree for all $z \neq 0$. 
5. Thus we need to compute \( \lim_{z \to 0} f(z) \).

6. If this limit exists then it should not depend on how we approach zero (theorem 4.2 in Rudin). In particular, we may approach zero via the real axis:

\[
\lim_{z \to 0} f(z) = \lim_{R \to 0} \frac{R}{|R|} = \lim_{R \to 0} R^{1-s}
\]

\( s > 1 \) is continuous

\[
\left( \lim_{R \to 0} R \right)^{1-s} = 0^{1-s} = 0
\]

where \( R \in (0, \infty) \)

7. Hence the limit exists, and thus if we define \( w = 0 \) then \( F \) is indeed continuous at 0 and we are set.

\[ \blacksquare \]

- This couldn’t have worked for \( s \geq 1 \) because then the limit \( \lim_{z \to 0} f(z) \) either diverges or does not exist.

### 1.2 Question 2

- **Claim:** \( f: \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C} \) defined by \( z \mapsto \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \) is continuous and \( f(z) = f(z+1) \).

**Note:** There is an identity saying that \( \pi \cot (\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \) but you are not supposed to know that.

**Proof:**

- Define the partial sums \( f_N(z) := \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2} \) for all \( N \in \mathbb{N} \).
- Define

\[
M_N := \sup \left\{ \left| f_N(z) - f(z) \right| \mid z \in \mathbb{C} \setminus \mathbb{Z} \right\}
\]

\[
= \sup \left\{ \left| \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2} - \pi \cot (\pi z) \right| \mid z \in \mathbb{C} \setminus \mathbb{Z} \right\}
\]

- We know that \( f_N \to f \) uniformly on \( \mathbb{C} \setminus \mathbb{Z} \) if and only if \( M_N \to 0 \) as \( N \to \infty \) (theorem 7.9 in Rudin).
- But \( M_N = \infty \) clearly, so that it does not converge to zero!
- Thus \( f_N \) cannot converge uniformly to \( f \), and we may not use uniform convergence to conclude continuity of \( f \).
- Instead, what you should have done is tried to prove uniform continuity on some subset of \( \mathbb{C} \setminus \mathbb{Z} \).
- Let \( z \in \mathbb{C} \setminus \mathbb{Z} \) be given, and pick some \( \varepsilon > 0 \) so that \( B_{\varepsilon}(z) = \{ \omega \in \mathbb{C} \mid |z - \omega| \leq \varepsilon \} \subseteq \mathbb{C} \setminus \mathbb{Z} \).
  - This is possible because \( (\mathbb{C} \setminus \mathbb{Z}) \in \text{Open} (\mathbb{C}) \) (because \( \mathbb{Z} \in \text{Closed} (\mathbb{C}) \) (because a singleton \( \{ z_0 \} \in \text{Closed} (\mathbb{C}) \) for all \( z_0 \in \mathbb{C} \) and \( \mathbb{Z} \) is a union of closed such singletons)).

  - **Claim:** \( f_N|_{B_{\varepsilon}(z)} \to f|_{B_{\varepsilon}(z)} \) uniformly.

**Proof:**

- Choose \( N_1 \in \mathbb{N} \) so that \( 2(|z| + \varepsilon) \leq N_1 \). Then for all \( N > N_1 \) we have

\[
M_N := \sup \left\{ \left| f_N|_{B_{\varepsilon}(z)}(w) - f|_{B_{\varepsilon}(z)}(w) \right| \mid w \in B_{\varepsilon}(z) \right\}
\]

\[
= \sup \left\{ \left| \sum_{n=N+1}^{\infty} \frac{2w}{w^2 - n^2} \right| \mid w \in B_{\varepsilon}(z) \right\}
\]

\[
\leq \sup \left\{ \left| \sum_{n=N+1}^{\infty} \frac{2w}{w^2 - n^2} \right| \mid w \in B_{\varepsilon}(z) \right\}
\]

\[
= \sup \left\{ \left| 2 \frac{|z| + \varepsilon}{n^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \right| \mid w \in B_{\varepsilon}(z) \right\}
\]

\[
\leq \sup \left\{ \left| 2 \frac{|z| + \varepsilon}{n^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \right| \mid w \in B_{\varepsilon}(z) \right\}
\]

\[
\to 0
\]
1.3 Question 3

- Let $A$ be some countable subset of $\mathbb{R}$, and let $\sum_{n=1}^{\infty} s_n$ be an absolutely convergent series of real numbers.

- Define $f(x) := \sum_{n=1}^{\infty} s_n \text{sign} (x - a_n)$ where

\[
\text{sign}(x) \equiv \begin{cases} 
1 & x > 0 \\
0 & x = 0 \\
-1 & x < 0
\end{cases}
\]

- **Claim:** The partial sums $f_N = \sum_{n=1}^{N} s_n \text{sign} (x - a_n)$ converge uniformly to $f$.

  **Proof:**

  - Use the Weierstrass $M$ test with $M_n \equiv s_n$.

- **Claim:** $f$ is continuous on $\mathbb{R} \setminus A$.

  **Proof:**

  - Follows from uniform convergence.

- **Claim:** $\lim_{\varepsilon \to 0} f (a_n + \varepsilon) - \lim_{\varepsilon \to 0} f (a_n - \varepsilon) = 2s_n$.

  **Proof:**

  - Make the calculation

\[
\lim_{\varepsilon \to 0} f (a_n + \varepsilon) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} f_N (a_n + \varepsilon) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \sum_{j=1}^{N} s_j \text{sign} (a_n + \varepsilon - a_j) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \left( s_n \text{sign} (\varepsilon) + \sum_{j=1}^{N} s_j \text{sign} (a_n + \varepsilon - a_j) \right) = s_n + \lim_{\varepsilon \to 0} \lim_{N \to \infty} \sum_{j=1}^{N} s_j \text{sign} (a_n + \varepsilon - a_j)
\]

  - In a very similar fashion we can calculate that $\lim_{\varepsilon \to 0} f (a_n - \varepsilon) = -s_n + P$.

  - Still need to show that $P$ exists to make this rigorous. Have a look in the official solutions for details.

- **Claim:** If $s_n > 0$ for all $n \in \mathbb{N}$ then $f$ is monotonically increasing.

  **Proof:**

  - The function $x \mapsto s_n \text{sign} (x - a_n)$ is monotonically increasing for any $n$ (homework).

  - The sum of monotone increasing functions is monotone increasing.

  - Due to $a_n \leq b_n \implies \lim a_n \leq \lim b_n$ we have that $f$ is monotonically increasing.

1.4 Question 4

- Almost everyone did it well. Just remember that you must define the domain of a function whenever you are defining a function.

1.5 Question 5

- Let $X$ and $Y$ be metric spaces, and let $(A_j)_{j=0}^{n-1} \subseteq \text{Closed} (X)$ for some $n \in \mathbb{N}$. Define $A := \bigcup_{j=0}^{n-1} A_j$.

- Part (a) **Claim:** $f : A \to Y$ is continuous if and only if $f|_{A_i} : A_i \to Y$ is continuous for all $i \in \mathbb{Z}_n$.

  **Proof:**

\[
\implies
\]

* Let $i \in \mathbb{Z}_n$.
* We know that \( f : A \to Y \) is continuous. Thus, \( \forall x \in A, \forall \varepsilon > 0 \exists \delta f (\varepsilon, x) > 0 \) such that if \( \tilde{x} \in B_{\delta f (\varepsilon, x)} (x) \) then \( f (\tilde{x}) \in B_{\varepsilon} (f (x)) \).

* Let \( \varepsilon > 0 \) be given, and let \( x \in A_i \) be given.

  * Take \( \delta f \lvert_{A_i} (x, \varepsilon) := \delta f (x, \varepsilon) \).

  * Then if \( \tilde{x} \in B_{\delta f \lvert_{A_i} (x, \varepsilon)} (x) \cap A_i \) then \( f (\tilde{x}) \in B_{\varepsilon} (f (x)) \) which implies \( f \lvert_{A_i} (\tilde{x}) \in B_{\varepsilon} (f \lvert_{A_i} (x)) \) because both \( x \) and \( \tilde{x} \) lie in \( A_i \).

\[ \begin{align*}
\text{Let } x & \in A \text{ and some } \varepsilon > 0 \text{ be given.} \\
\text{Define } I & := \{ i \in \mathbb{Z}_n \mid x \in A_i \}. \\
\text{Define } f \lvert_{A_i} & \text{ is continuous at } x \text{ for all } i \in I. \\
\text{Then if } \tilde{x} & \in B_{\delta f \lvert_{A_i} (x, \varepsilon)} (x) \cap A_i \text{ then } f \lvert_{A_i} (\tilde{x}) \in B_{\varepsilon} (f \lvert_{A_i} (x)) \text{ for all } i \in I \text{ (there exist such } \delta f \lvert_{A_i} (x, \varepsilon)). \\
\text{From this it follows that if } \tilde{x} & \in B_{\delta f \lvert_{A_i} (x, \varepsilon)} (x) \cap A_i \text{ then } f (\tilde{x}) \in B_{\varepsilon} (f (x)) \text{ for all } i \in I \text{ (there exist such } \delta f \lvert_{A_i} (x, \varepsilon)).
\end{align*} \]

* Define \( \tilde{\delta} (x, \varepsilon) := \min \left( \left\{ \delta f \lvert_{A_i} (x, \varepsilon) \mid i \in I \right\} \right) \).

* Then if \( \tilde{x} \in B_{\tilde{\delta} (x, \varepsilon)} (x) \cap (\bigcup_{i \in I} A_i) \) then \( f (\tilde{x}) \in B_{\varepsilon} (f (x)) \).

* Define \( J := \mathbb{Z}_n \setminus I \).

* Define \( C := \bigcup_{i \in J} A_i \).

* Claim: \( C \in \text{Closed} (X) \).

  * \( C \) is a finite union of closed subsets of \( X \). The property of being closed is “closed” under finite unions.

* Claim: \( x \notin C \).

  * \text{Proof:}

    * Thus \( (X \setminus C) \in \text{Open} (X) \) such that \( x \in (X \setminus C) \).

    * Thus, \( \exists \tilde{\delta} (x, \varepsilon) > 0 \) such that \( B_{\tilde{\delta} (x, \varepsilon)} (x) \subseteq (X \setminus C) \).

    * Thus, \( B_{\tilde{\delta} (x, \varepsilon)} (x) \cap C = \emptyset \).

    * Define \( \delta (x, \varepsilon) := \min \left( \left\{ \tilde{\delta} (x, \varepsilon), \tilde{\delta} (x, \varepsilon) \right\} \right) \).

    * Thus if \( \tilde{x} \in B_{\delta (x, \varepsilon)} (x) \), then \( \tilde{x} \notin C \) and so \( \tilde{x} \in \bigcup_{i \in I} A_i \), and \( x \in B_{\delta (x, \varepsilon)} (x) \) so that we may conclude \( f (\tilde{x}) \in B_{\varepsilon} (f (x)) \).

  * For part (b):

    - Define \( A_0 = [0, \infty) \) and \( A_1 = (-\infty, 0) \), and define \( f \lvert_{A_0} := (x \mapsto 1) \) and \( f \lvert_{A_1} := (x \mapsto 0) \). Then define \( f : \mathbb{R} \to \mathbb{R} \) as in (a), where \( A_0 \cup A_1 = \mathbb{R} \).

    - Because the restrictions \( f \lvert_{A_0} \) and \( f \lvert_{A_1} \) are constant they are continuous, yet, \( f \) is not continuous at 0.4

1.6 Question 6

* This was largely covered in the colloquium on the Cantor set. You may read the summary of that colloquium and also the official solutions to the exercises.

2 Exercise Sheet Number 10

2.1 Differentiation

Let \( f : [a, b] \to \mathbb{R} \) be a function. Then for any \( x \in [a, b] \) define

\[
 f' (x) \equiv \lim_{{t \to x}} \frac{{f(t) - f(x)}}{{t - x}}
\]

if the limit exists.

* If the limit exists, we say that \( f \) is differentiable at \( x \), and that \( f' \) is its derivative at \( x \).

* Claim: If \( f \) is differentiable at \( x \in [a, b] \) then \( f \) is continuous at \( x \).

  * \text{Proof:}


limit characterization of continuity:
\[
\lim_{t \to x} f(t) = \lim_{t \to x} [f(t) - f(x) + f(x)] \\
= \lim_{t \to x} \left[ \frac{f(t) - f(x)}{t - x} (t - x) + f(x) \right] \\
= \lim_{t \to x} \left[ \frac{f(t) - f(x)}{t - x} (t - x) \right] + f(x) \\
= \left[ \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \right] \left[ \lim_{t \to x} (t - x) \right] + f(x) \\
= f'(x) \cdot 0 + f(x) \\
= f(x)
\]

- The converse of this theorem is false! (Think about \( x \to |x| \) at 0).
- Example: Define \( f: \mathbb{R} \to \mathbb{R} \) by \( x \mapsto x^2 \). Then

\[
f'(x) = \lim_{t \to x} \frac{t^2 - x^2}{t - x} \\
= \lim_{t \to x} (t + x) \\
= 2x
\]

### 2.2 Concrete Tips for the Homework Exercises

#### 2.2.1 Question 1
- for part (a) use the binomial formula on \([\cos(x)]^n = \left[ e^{ix} + e^{-ix} \right]^n\)
- For part (b) use
  1. induction
  2. the identity \( \cos((n+1)x) = 2\cos(nx)\cos(x) + \cos((n-1)x) \) (which you can verify easily).

#### 2.2.2 Question 2
- Calculate \( \lim_{x \to \pm \pi/2} \tan(x) \) (from above or from below, depending on whether the plus or minus signs are chosen).
- Use the intermediate value theorem.

#### 2.2.3 Question 3
- Use induction together with:
  1. the “ordinary” Leibniz rule.
  2. the fact that \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \)

#### 2.2.4 Question 4
- May not use the intermediate value theorem, because \( f' \) is not necessarily continuous!

#### 2.2.5 Question 5
- Define \( \forall k \in \mathbb{Z}_n \equiv \{ 0, \ldots, n-1 \} \ f_k(x) := \left[ (1 - x^2)^n \right]^{(k)}. \)
- Then \( P_n(x) = \frac{1}{2^m} f_n(x). \)
- Show that \( f_k(-1) = 0 = f_k(1). \)
- For part (b):
  - Define \( f(x) := (x^2 - 1) p'(x) \) where \( p(x) := (x^2 - 1)^n. \)
  - Compute \( f^{(n+1)}(x) \) once with \( f(x) = (x^2 - 1) p'(x) \) and once with \( f(x) = 2nx p(x) \), and subtract the two equations you get.
  - Multiply by ... to get the desired equation.
  - Use question 3 (a).
2.2.6 Question 6

- Compute \( \lim_{x \to \pm \infty} f(x) \).
- Show that \( f'(x) > 0 \) for all \( x \).
- Compute \( f''(x) \) and conclude where \( f \) is concave and where it is convex.

2.2.7 Question 7

- For \( t = 0 \) you must compute the derivative by the actual definition.
- Show \( f' \) is not continuous at 0.
- Define \( t_k := \frac{1}{(2k+1)\pi} \) and show that \( \lim_{k \to \infty} t_k = 0 \) and \( f'(t_k) = 3 \) for all \( k \in \mathbb{N} \).