Analysis 1
Recitation Session of Week 11

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1 Exercise Sheet Number 9

1.1 Question 1

1.2 Question 2

- Be aware of the fact that $\Re\left(\frac{a+ib}{c+id}\right) \neq \frac{a}{c}$! In fact,

$$\begin{aligned}
\Re\left(\frac{a+ib}{c+id}\right) &= \Re\left(\frac{(a+ib)(c-id)}{c^2+d^2}\right) \\
&= \Re\left(\frac{ac+bd+i(bc-ad)}{c^2+d^2}\right) \\
&= \frac{ac+bd}{c^2+d^2} \\
&\neq \frac{a}{c}
\end{aligned}$$

  and simiarly for $\Im\left(\frac{a+ib}{c+id}\right)$.

- So that we have

$$\begin{aligned}
\sum_{j=0}^{n} \cos(jx) &= \sum_{j=0}^{n} \Re\left(e^{ijx}\right) \\
&= \Re\left(\sum_{j=0}^{n} e^{ijx}\right) \\
&= \Re\left(\sum_{j=0}^{n} \left(e^{ix}\right)^j\right) \\
&= \Re\left(\frac{1-\left(e^{ix}\right)^{n+1}}{1-e^{ix}}\right) \\
&= \Re\left(\frac{1-\left(e^{ix}\right)^{n+1}}{1-e^{ix}}\frac{1-e^{-ix}}{1-e^{-ix}}\right) \\
&= \Re\left(\frac{1-e^{i(n+1)x}-e^{-ix}+e^{inx}}{2-e^{ix}-e^{-ix}}\right) \\
&= \frac{\Re\left(1-e^{i(n+1)x}-e^{-ix}+e^{inx}\right)}{2(1-\cos(x))} \\
&= \frac{1-\cos((n+1)x)-\cos(x)+\cos(nx)}{2(1-\cos(x))}
\end{aligned}$$

1.3 Question 3

- Why can we exchange limit with series when proving $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$? Because the series of $\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ converges uniformly. See theorem 7.11 in Rudin.

1.4 Question 4

- Part (b):
- Assume K is compact.
- Assume $f : K -> Y$ is not uniformly continuous.
- *Claim*: f is not continuous.
  Proof:
  - Using (a) define two new sequences a_n and b_n such that $d_X(a_n, b_n) < 1/n$ yet $d_Y(f(a_n), f(b_n)) >= \epsilon_0$, for some ϵ_0 and for all n.
  - Because K is compact, there is a subsequence A of a_n converging to a and there is a subsequence B of b_n converging to b.
  - Claim: $a = b$.
    Proof:
    * Follows from the fact that $d_X(a_n, b_n) < \frac{1}{n}$.
  - Assume f is continuous.
  - Then $f(a_n) \to f(a)$ and $f(b_n) \to f(a)$ in the correspodning subsequences.
  - Thus, $d_Y(f(a_n), f(b_n)) \leqslant d_Y(f(a_n), f(a)) + d_Y(f(a), f(b_n))$ can be made arbitrarily small, yet we know that $d_Y(f(a_n), f(b_n)) \geqslant \epsilon_0$, hence, a contradiction.

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1.5 Question 5

- No balls or sequences necessary! The basic necessary facts are that $A \in \text{Closed}(X)$ iff $(X \setminus A) \in \text{Open}(X)$ and the "axioms" of topology (for metric spaces they are theorems, or rather, it is a theorem that the metric induces a bona fide topology).
- Let (X, d) be a metric space.
- *Claim*: $\varnothing \in \text{Closed}(X)$.
  Proof:
  - $\varnothing = X \setminus X$ and $X \in \text{Open}(X)$, so $\varnothing$ is the complement of an open set and as such it is closed.
- *Claim*: $X \in \text{Closed}(X)$.
  Proof:
  - $X = X \setminus \varnothing$ and $\varnothing \in \text{Open}(X)$.
- *Claim*: If $A_i \in \text{Closed}(X)$ for all $i \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ then $\left(\bigcup_{i=1}^{n} A_i\right) \in \text{Closed}(X)$.
  Proof:
  - $X \setminus A_i \in \text{Open}(X)$ because $A_i \in \text{Closed}(X)$.
  - Thus using the fact that a topology is closed under finite intersections, we have $\left(\bigcap_{i=1}^{n} (X \setminus A_i)\right) \in \text{Open}(X)$.
  - But using de Morgan's laws, we have that $\bigcap_{i=1}^{n} (X \setminus A_i) = \left(X \setminus \left(\bigcup_{i=1}^{n} A_i\right)\right)$, which implies $\left(X \setminus \left(\bigcup_{i=1}^{n} A_i\right)\right) \in \text{Open}(X)$ or $\left(\bigcup_{i=1}^{n} A_i\right) \in \text{Closed}(X)$.
- *Claim*: If $A_i \in \text{Closed}(X)$ for all $i \in I$ where I is an arbitrary indexing set, then $\left(\bigcap_{i \in I} A_i\right) \in \text{Closed}(X)$.
  Proof:
  - $(X \setminus A_i) \in \text{Open}(X)$ for all $i \in I$, so that $\left(\bigcup_{i \in I} (X \setminus A_i)\right) \in \text{Open}(X)$ because a topology is closed under arbitrary unions.
  - But using de Morgan's laws, we have that $\bigcup_{i \in I} (X \setminus A_i) = X \setminus \left(\bigcap_{i \in I} A_i\right)$.
  - Thus, $\left(X \setminus \left(\bigcap_{i \in I} A_i\right)\right) \in \text{Open}(X)$ or rather $\left(\bigcap_{i \in I} A_i\right) \in \text{Closed}(X)$ as desired.
- Observe that the Cantor set is closed because it is the arbitrary intersection of closed intervals.

1.6 Question 6

- The point of this exercise is that open sets (which we naturally think of as open *intervals* can be not so simple. Yet we can always decompose them in a simple way. If you want an example of a not-so-simple open set, think of $[0, 1]\setminus\mathtt{Cantor}$.
- Let $U \in \mathtt{Open}(\mathbb{R})$.
- Part (a): *Claim*: $\forall x \in U \exists (a_x, b_x) \in (\{-\infty\} \cup (\mathbb{R}\setminus U)) \times (\{\infty\} \cup (\mathbb{R})\setminus U)$ such that $x \in (a_x, b_x) \subseteq U$.
  Proof:
  - Define $\begin{cases} a_X & := \sup((-\infty, x)\setminus U) \\ b_x & := \inf((x, \infty)\setminus U) \end{cases}$ with the convention that $\sup(\varnothing) \equiv -\infty$ and $\inf(\varnothing) \equiv \infty$.
  - *Claim*: $a_x < x < b_x$.
    Proof:
    * $U \in \mathtt{Open}(\mathbb{R})$, so $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$, where $B_\varepsilon(x) \equiv (x-\varepsilon, x+\varepsilon)$.
    * From this it follows that $((-\infty, x)\setminus U) \subseteq (-\infty, x-\varepsilon)$ and that $((-\infty, x)\setminus U) \subseteq (x+\varepsilon, \infty)$.
    * But if $A \subseteq B$ then $\sup(A) \leqslant \sup(B)$ and $\inf(A) \geqslant \inf(B)$.
    * Thus $\underbrace{\sup((-\infty, x)\setminus U)}_{a_x} \leqslant \underbrace{\sup((-\infty, x-\varepsilon))}_{x-\varepsilon<x}$, and similarly, $b_x \geqslant x+\varepsilon > x$.
  - *Claim*: $(a_x, b_x) \subseteq U$.
    Proof:
    * Assume otherwise. Thus, $\exists t \in (a_x, b_x)\setminus U$.
    * But $x \in U$, so that $t \neq x$, that is, $x < t$ or $t < x$.
    * Case 1: $t \in (a_x, x)$.
      · Recall that $a_x \equiv \sup((-\infty, x)\setminus U)$, so that we have a contradiction, as $t \in (-\infty, x)\setminus U$ and $a_x < t$, contradicting a_x being the supremum of that set.
    * Case 2: $t \in (x, b_x)$.
      · This contradicts b_x being the infimum of $(-\infty, x)\setminus U$, as $t \in (-\infty, x)\setminus U$ yet $t < b_x$.
  - *Claim*: $a_x \in (\{-\infty\} \cup (\mathbb{R}\setminus U))$.
    Proof:
    * If $a_x = -\infty$ we are done.
    * Otherwise, assume $a_x \in U$.
    * But $U \in \mathtt{Open}(\mathbb{R})$, so that $\exists \varepsilon > 0$ such that $B_\varepsilon(a_x) \subseteq U$.
    * However, $(-\infty, x)\setminus U \subseteq (-\infty, a_x - \varepsilon)$ as $B_\varepsilon(a_x) \subseteq U$.
    * Thus we have that $a_x \equiv \sup((-\infty, x)\setminus U) \leqslant \sup((-\infty, a_x - \varepsilon)) = a_x - \varepsilon < a_x$.
    * In particular, $a_x \neq a_x$, which is a contradiction.
  - *Claim*: $b_x \in (\{\infty\} \cup (\mathbb{R}\setminus U))$.
    Proof:
    * Analogously to the preceding proof.

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- Define $I_x := (a_x, b_x)$ where a_x and b_x are as defined above, $\forall x \in U$.
- *Claim*: $\forall (x, y) \in U^2$, either $I_x = I_y$ or $I_x \cap I_y = \varnothing$.
  Proof:
  - Let $(x, y) \in U^2$ be given.
  - *Claim*: If $y \in I_x$ then $I_x \subseteq I_y$.
    Proof:
    * Assume $y \in I_x$.
    * *Claim*: $a_y \leqslant a_x$.
    * *Proof*:
      · Case 1: $a_x = -\infty$.
      1. Because $\left(\underbrace{a_x}_{-\infty}, b_x\right) \subseteq U$, we cannot find any element to the left of U which is larger than $-\infty$.
      2. But $a_y \notin U$ and has to be to the left of it, so that necessarily $a_y = -\infty$.
      · Case 2: $a_x > -\infty$.
      1. Recall that $a_y \equiv \inf((-\infty, y)\setminus U)$ and clearly from $y \in (a_x, b_x)$ we have $a_x > y$, but $a_x \notin U$, so that $a_x \in (-\infty, y)\setminus U$. As a result, $a_y \leqslant a_x$.

    * Thus in either case $a_y \leqslant a_x$. Very similarly we can prove that $b_x \leqslant b_y$.
    * Thus, $I_x \subseteq I_y$.
  - However, in the proof, x and y's roles were symmetric. So we can apply the very same proof exchanging x and y to obtain in addition that if $x \in I_y$ then $I_y \subseteq I_x$.
  - If $I_x \cap I_y = \varnothing$ we are done.
  - Otherwise, $\exists z \in I_x \cap I_y$.
  - Because $I_x \subseteq U$, $z \in U$ and so we may apply the previous proof to obtain that $I_x \subseteq I_z$ and $I_y \subseteq I_z$.
  - Then $(x, y) \in {I_z}^2$.
  - Applying the previous proof again we obtain that $I_z \subseteq I_x$ and $I_z \subseteq I_y$.
  - As a result, $I_x = I_y$.

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- *Claim*: $\exists A \subseteq U$ such that $U = \bigcup_{x \in A} I_x$ and $|I_x| = |\mathbb{N}|$, and $I_x \cap I_y = \varnothing$ for all $(x, y) \in A^2$ such that $x \neq y$.
  Proof:
  - Define $\mathcal{I} := \{ I_x \mid x \in U \}$.
  - It is then clear that $\bigcup \mathcal{I} = U$ because each element $I_x \in \mathcal{I}$ stems from some $x \in U$, and so we cover the whole of U (or even more).
  - We will construct a map $g : \mathcal{I} \to B$ such that $B \subseteq \mathbb{Q}$ and g is a bijection. This will prove our claim together with the fact that using the preceding proof, all the (distinct) elements in $\mathcal{I}$ are disjoint.
  - Define $\hat{\mathcal{I}} := \{ I \cap \mathbb{Q} \mid I \in \mathcal{I} \}$. The elements of $\hat{\mathcal{I}}$ are not empty as every interval contains rational numbers.
  - Define a map $f : \hat{\mathcal{I}} \to \mathbb{Q}$ by picking one element out of $I \cap \mathbb{Q}$ (this is possible due to the axiom of choice).
  - Now define a map $g : \mathcal{I} \to \mathbb{Q}$ by $g(I) := f(I \cap \mathbb{Q})$.
  - Because f was defined to be a point inside of $I \cap \mathbb{Q}$, we then have that $g(I) \in I \cap \mathbb{Q}$.
  - *Claim*: g is injective.
    Proof:
    * Assume otherwise. Then $\exists (I, I') \in \mathcal{I}^2$ such that $I \neq I'$ and $g(I) = g(I')$. But $g(I) \in I$ by definition, and so $g(I') \in I$. As a result of the preceding proof we have that either $I \cap I' = \varnothing$ or $I = I'$. But $g(I) \in I \cap I'$ and so $I \cap I' \neq \varnothing$. In particular, $I = I'$. This is then a contradiction.
  - Using the preceding proof, we have that $I = I_{g(I)}$.
  - Thus, for every $I \in \mathcal{I}$, we may find some $q \in \mathrm{im}(g)$ such that $I = I_q$.
  - But $\mathrm{im}(g) \subseteq \mathbb{Q}$, $|\mathrm{im}(g)| \leqslant |\mathbb{N}|$.
  - In addition, we clearly have $U = \bigcup_{q \in \mathrm{im}(g)} I_q$ because g is surjective onto $\mathrm{im}(g)$.

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2 Exercise Sheet Number 11

2.1 The Hospital's Rule

- Let (a, b) be a given interval in $\mathbb{R}$, where we allow for either $a = -\infty$ or $b = \infty$.
- Let $(f, g) \in \mathbb{R}^{\mathbb{R}}$ be two differentiable functions (at least in (a, b)) and assume further that $0 \notin g'((a, b))$ and that $\lim_{x \to a} \left(\frac{f'(x)}{g'(x)} \right)$ exists.
- *Claim*: If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or if $\lim_{x \to a} g(x) = \pm\infty$ then

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to a} \left(\frac{f'(x)}{g'(x)} \right)$$

- The same claim is true in $x \to b$ case.
- Example:
  - To compute $\lim_{x \to 0} \frac{\sin(x)}{x}$, we note that $\lim_{x \to 0} \frac{\cos(x)}{1}$ exists (it is 1) and that both nominator and denominator go to zero as $x \to 0$.
  - Don't always use it: $\lim_{x \to 0} \frac{\sin(x^3)}{\sin^3(x)} = \lim_{x \to 0} \frac{\sin(x^3)}{\sin^3(x)} \frac{x^3}{x^3} = \lim_{x \to 0} \left(\frac{\sin(x^3)}{x^3} \right) \left(\frac{x}{\sin(x)} \right)^3 = 1 \cdot 1^3 = 1$.
  - Cannot apply it on $\lim_{x \to \infty} \frac{x + \sin(x)}{x + \cos(x)} = \frac{\infty}{\infty}$ because $\lim_{x \to \infty} \frac{1 + \cos(x)}{1 - \sin(x)}$ does not exist.

2.2 Smooth Functions (Glatte Funktionen)

- Define C^0 to be the subset of $\mathbb{R}^{\mathbb{R}}$ where each element is continuous.
- Define C^1 to be the subset of $\mathbb{R}^{\mathbb{R}}$ where each element is differentiable (and is thus continuous) and has a continuous derivative. That is, whose derivative is in C^0. This is called continuously differentiable.
- Inductively then, for each $k \in \mathbb{N}\backslash\{0\}$, a C^k function is a differentiable function whose derivative is in C^{k-1}.
- Clearly we have $C^k \subseteq C^{k-1}$ for all $k \in \mathbb{N}$. There are examples that show in fact that $C^k \subsetneq C^{k-1}$.
- Smooth maps are maps in C^∞.
- Examples:
  - $f(x) = \begin{cases} x & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$ is in C^0 but not in C^1.
  - $f(x) = |x|^{k+1}$ where $k \in 2\mathbb{Z}$ is continuous and k-times differentiable. At $x = 0$ they are not $(k+1)$-differentiable, so they are in C^k but not in C^j for all $j > k$.
  - $\exp(x)$ is in C^∞.
  - $f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & otherwise \end{cases}$ is in C^∞.

2.3 Taylor Series

- The Taylor series of f around the point a is given by (Theorem 8.4 in Rudin):

$$\begin{aligned} f(x+a) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(f^{(n)}(a)\right)(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \ldots \end{aligned}$$

- Not *all* functions have such a power series representation, but if they do, the representation is unique and allows very convenient manipulations and approximations.
- If you go only up to mth order in x, then the truncated polynomial is called the m-th Taylor polynomial of f.

2.4 A Few Tips

2.4.1 Question 5

- Use the fact that $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence radius 1.
- Look at theorem 7.17 in Rudin. In particular, if we have uniform convergence, and if the derivatives of the partial sums converge uniformly as well, then we may differentiate term by term.
- Compute $xf'(x)$.
- Compute $xf'(x) + x^2 f''(x)$.
- Plug in $x = \frac{1}{2}$ in the above.