Analysis 1
Recitation Session of Week 12

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1 Exercise Sheet Number 10

1.1 Question 4

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function.

- *Claim*: For all γ between $f'(a)$ and $f'(b)$ there exists a $c \in [a, b]$ such that $f'(c) = \gamma$.
  Proof:
  - If $f'(a) = \gamma$ or $f'(b) = \gamma$ then we are done.
  - Otherwise, define $g : [a, b] \to \mathbb{R}$ by $x \mapsto f(x) - \gamma x$.
  - Then $g'(x) = f'(x) - \gamma$. If we could find a point c in $[a, b]$ such that $g'(c) = 0$ then we'd be done.
  - Because $f'(a) \neq \gamma$ and $f'(b) \neq \gamma$, $g'(a) \neq 0$ and $g'(b) \neq 0$.
  - We cannot have $g'(a) = g'(b)$ because then $f'(a) = f'(b)$ and then there are no points between $f'(a)$ and $f'(b)$ and the statement is vacuously fulfilled.
  - So then we have either that $g'(a) < 0 < g'(b)$ or that $g'(a) > 0 > g'(b)$.
  - Assume conversely that g has *no* extremum. Then g is monotone.
  - If g is not *strictly* monotone, for example, if $\exists$ some $x < y$ such that $g(x) = g(y)$ then using Rolle's theorem we have some $z \in (x, y)$ such that $g'(z) = 0$. [Rolle's theorem says: If a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.]
  - Otherwise we assume that g is *strictly* monotone (increasing or decreasing).
  - Then $\frac{g(x)-g(y)}{x-y} \geqslant 0$ (or $\leqslant 0$) for all $(x, y) \in [a, b]^2$.
  - If we take the limit of $y \to x$ we get that $g'(x) \geqslant 0$ (or $g'(x) \leqslant 0$) for all $x \in [a, b]$.
  - But this contradicts the fact that $g'(a)$ and $g'(b)$ have opposite signs.

2 Exercise Sheet Number 12

Note: This exercise sheet is *not to be handed in* because of the midterm that is taking place during the 13th recitation session next week.

2.1 The Riemann Integral

We follow Rudin's *PMA*.

- Let $[a, b]$ be a given interval.
- Let $f : [a, b] \to \mathbb{R}$ be bounded.

2.1.1 Partitions of a Closed Interval

- A partition P of $[a, b]$ is a finite sequence of points $P = (x_0, x_1, \ldots, x_n) \in [a, b]^{n+1}$ such that

$$a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1} \leqslant x_n = b$$

- For convenience define $\Delta x_i := x_i - x_{i-1}$ for all $i \in \{1, \ldots, n\}$.

2.1.2 The Upper and Lower Sums

- For each partition P of $[a, b]$, define
$$\begin{cases} M_i^P := \sup\left(\{ f(x) \mid x_{i-1} \leqslant x \leqslant x_i \}\right) \\ m_i^P := \inf\left(\{ f(x) \mid x_{i-1} \leqslant x \leqslant x_i \}\right) \end{cases}$$
- And then
$$\begin{cases} U(P, f) & := \sum_{i=1}^{n} M_i^P \Delta x_i \\ L(P, f) & := \sum_{i=1}^{n} m_i^P \Delta x_i \end{cases}$$

2.1.3 The Upper and Lower Riemann Integrals

- Define
$$\begin{cases} U\int_a^b f dx & := \inf\left(\{ U(P, f) \mid P \text{ is a partition of } [a, b] \}\right) \\ L\int_a^b f dx & := \sup\left(\{ L(P, f) \mid P \text{ is a partition of } [a, b] \}\right) \end{cases}$$
- Note that f is bounded, so that $\exists (m, M) \in \mathbb{R}^2$ such that $m \leqslant f(x) \leqslant M \forall x \in [a, b]$.
- As such, for any partition P of $[a, b]$, we have the following relation
$$m(b-a) \leqslant L(P, f) \leqslant U(P, f) \leqslant M(b-a)$$
- Taking the infimum, or supremum of this relation shows immediately that the upper and lower Riemann integrals are always finite, as such defined, for any bounded f!
- The question of whether they are equal or not is the hard question to answer.

2.1.4 The Riemann Integral

- For f, if $U\int_a^b f dx = L\int_a^b f dx$, we say that f is Riemann integrable, and define its Riemann integral to be $L\int_a^b f dx = U\int_a^b f dx$, which we denote simply as $\int_a^b f dx$.

2.1.5 Some Remarks

- *Claim*: $\int_0^1 x dx = \frac{1}{2}$.
  Proof:
  - For each partition P of $[a, b]$, we have
$$\begin{cases} M_i^P \equiv \sup\left(\{ x \mid x_{i-1} \leqslant x \leqslant x_i \}\right) & = x_i \\ m_i^P \equiv \inf\left(\{ x \mid x_{i-1} \leqslant x \leqslant x_i \}\right) & = x_{i-1} \end{cases}$$
  - And then
$$\begin{cases} U(P, f) & := \sum_{i=1}^{n} x_i \Delta x_i = \sum_{i=1}^{n} x_i (x_i - x_{i-1}) \\ L(P, f) & := \sum_{i=1}^{n} x_{i-1} \Delta x_i = \sum_{i=1}^{n} x_{i-1} (x_i - x_{i-1}) \end{cases}$$
  - In order to compute the infimum and supremum of these things, we need a systematic way to generate partitions. We can define $P_n : \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right)$.
  - Then one can show that an infimum *or* supremum over all possible partitions will be equal to the value of the "partition" P_∞. This is because, certainly P_n will be in the set of all partitions for each n, yet, for any set of partitions you could think of, there is some n large enough that is not contained in it, and so by examining P_n of that n, the supremum will become larger and the infimum will become smaller. (See theorem 6.4 in Rudin about refinements of partitions. In particular, if P^* is a refinement of P then $L(P, f) \leqslant L(P^*, f)$ and $U(P, f) \geqslant U(P^*, f)$.

– Thus we can write that (using $x_i = \frac{i}{n}$):

$$\begin{aligned} L\int_0^1 f dx &= \lim_{n\to\infty} L(P_n, f) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n}\left(\frac{i}{n} - \frac{i-1}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} \\ &= \lim_{n\to\infty} \frac{1}{n^2} \underbrace{\sum_{i=1}^{n} i}_{\frac{1}{2}n(n+1)} \\ &= \frac{1}{2}\underbrace{\lim_{n\to\infty} \frac{n(n+1)}{n^2}}_{1} \end{aligned}$$

– Similarly you can show that $U\int_0^1 f dx = \frac{1}{2}$ and so f is Riemann integrable on $[0, 1]$ with its integral being equal to $\frac{1}{2}$.

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- Usually, however, you wouldn't compute integrals like the above procedure!
- You would rather use "guesses" using the fundamental theorem of calculus:
- Theorem 6.20 in Rudin:
  – Let f be Riemann integrable on $[a, b]$ and define a new map $F : [a, b] \to \mathbb{R}$ by $x \mapsto \int_a^x f(t)\,dt$.
  – Then F is continuous on $[a, b]$ and if f is continuous at $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.
  – Thus, if you can find a differentiable function whose derivative at a point gives you the integrand of the integral you are looking for, you have solved the integral.
  – In such ways you can make nice guesses for the result of integrals without computing the Riemann upper and lower integrals explicitly.
- Every continuous function on $[a, b]$ is Riemann-integrable on $[a, b]$. (Theorem 6.8 in Rudin). However, there are examples of functions which are not continuous and are still Riemann integrable:
  – Let $f : [-1, 1] \to \mathbb{R}$ be defined by $x \mapsto \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$.
  – It is clear that f is not continuous at 0.
  – However, *Claim*: f is Riemann integrable.
  Proof:
    * The lower Riemann sums will always give zero, no matter what partition we take, so that we can immediately write $L(P, f) = 0$ for any P, and so $L\int_{-1}^{1} f dx = 0$.
    * The upper Riemann sum of some partition P will be of the form Δx_i where i is the partition that contains the point 0. But as we take the infimum over all such partitions, the length Δx_i becomes small and small and its limit value is zero, so that the upper Riemann integral is also zero!
    * As such the upper and lower Riemann integrals are equal and so the function is Riemann integrable (with its integral being equal to 0).

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- In fact later you will see that there are many non-continuous functions which are still Riemann-integrable.
- A summary of strategies to solve integrals:
  1. Memorize the following:
     (a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$.
     (b) $\int \frac{1}{x} dx = \log(|x|) + C$
     (c) $\int e^x dx = e^x + C$
     (d) $\int \sin(x)\,dx = -\cos(x) + C$
     (e) $\int \cos(x)\,dx = \sin(x) + C$
     (f) $\int \frac{1}{[\cos(x)]^2} dx = \tan(x) + C$

(g) $\int \frac{\sin(x)}{[\cos(x)]^2} dx = \frac{1}{\cos(x)} + C$

(h) $\int \frac{1}{1+x^2} dx = \arctan(x) + C$

(i) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$

2. Substitution

(a) $\int_a^b f(x)\, dx$ is given.

(b) Define t by $x = g(t)$, where g is a differentiable and invertible function, with inverse $g^{-1}(x)$.

(c) Then $\int_a^b f(x)\, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\, g'(t)\, dt$.

(d) Example:

$$\begin{aligned}\int 2x\cos\left(x^2\right) dx &\overset{y=x^2}{=} \int \cos(y)\, dy\\ &= \sin(y)\\ &= \sin\left(x^2\right)\end{aligned}$$

3. Integration by parts:

(a) $\int_a^b f'(x)\, g(x)\, dx = f(x)\, g(x)|_a^b - \int_a^b f(x)\, g'(x)\, dx$.

(b) A nice way to remember this is that this is essentially the reverse of the Leibnitz rule: $\int (f'g + fg') = \int (fg)' = fg$.

(c) Example of usage:

$$\begin{aligned}\int \frac{\log(x)}{x^2}\, dx &= \int -\underbrace{\left(-\frac{1}{x^2}\right)}_{f'} \underbrace{\log(x)}_{g} dx\\ &= \int -\left(\frac{1}{x}\right)' \log(x)\, dx\\ &= -\frac{1}{x}\log(x) + \int \frac{1}{x}\log'(x)\, dx\\ &= -\frac{1}{x}\log(x) + \int \frac{1}{x}\frac{1}{x} dx\\ &= -\frac{1}{x}\log(x) + \int \frac{1}{x^2} dx\\ &= -\frac{1}{x}\log(x) - \frac{1}{x}\end{aligned}$$

4. Trigonometric Identities:

(a) Use trigonometric identities to reduce the powers of trigonometric functions. Reducing the powers will help you solve the integral: $[\sin(x)]^2 = \frac{1}{2}[1 - \cos(2x)]$.

(b) Example:

$$\begin{aligned}\int [\sin(x)]^5\, dx &= \int \sin(x)\,[\sin(x)]^4\, dx\\ &= \int \sin(x)\left[(\sin(x))^2\right]^2 dx\\ &= \int \sin(x)\,[1-\cos(x)]^2\, dx\\ &\overset{u=\cos(x)}{=} \int -\left(1-u^2\right)^2 du\\ &= \int -\left(1-2u^2+u^4\right) du\\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5\\ &= -\cos(x) + \frac{2}{3}[\cos(x)]^3 - \frac{1}{5}[\cos(x)]^5\end{aligned}$$

5. Trigonometric Substritution:

(a) Example:

$$\begin{aligned}
\int \sqrt{1-x^2}dx &\overset{u=\arcsin(x)}{=} \int \sqrt{1-[\sin(u)]^2}\cos(u)\,du \\
&= \int \sqrt{[\cos(u)]^2}\cos(u)\,du \\
&= \int [\cos(u)]^2\,du \\
&= \int \frac{1+\cos(2u)}{2}du \\
&= \frac{1}{2}u + \frac{1}{4}\sin(2u) \\
&= \frac{1}{2}\arcsin(x) + \frac{1}{4}\underbrace{\sin(2\arcsin(x))}_{2x\sqrt{1-x^2}}
\end{aligned}$$

6. Rational Functions:
   (a) Rational functions are functions which looks like $\frac{p(x)}{q(x)}$ where both $p(x)$ and $q(x)$ are polynomials.
   (b) In general there is a technique called "partial fractions" which in principle allows us to integrate any such rational function.
   (c) For example: $\frac{7-6}{(x-2)(x+3)} = \frac{8}{5}\frac{1}{x-2} + \frac{27}{5}\frac{1}{x+3}$.
7. Use a computer.

2.2 Question 1

- Notation: recall that $C^0([a,b],\mathbb{R})$ is the set of all continuous functions from $[a,b]\to\mathbb{R}$.

2.2.1 Recall the Inner Product

Recall from Week 5 (this definition is tailored now for the question at hand, and is *not* the most general):

A (real) inner product space is a tuple $\left(C^0([a,b],\mathbb{R}), \langle\cdot,\cdot\rangle\right)$ where $C^0([a,b],\mathbb{R})$ is a vector space over a field $\mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is a map from $C^0([a,b],\mathbb{R})^2 \to \mathbb{R}$ such that:

1. $\langle u, v\rangle = \langle v, u\rangle$ for all $(u,v)\in C^0([a,b],\mathbb{R})^2$.
2. $\langle u, u\rangle \geqslant 0$ for all $u \in C^0([a,b],\mathbb{R})$.
3. $\langle u, u\rangle = 0 \iff u = 0_{C^0([a,b],\mathbb{R})}$.
4. $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ for all $(\alpha,\beta,u,v,w)\in\mathbb{R}^2\times C^0([a,b],\mathbb{R})^3$.

If you want to be nice you would also show that $C^0([a,b],\mathbb{R})$ is a vector space.

Thus, to solve this question, you must prove that the Riemann integral obeys all four of these properties. You may use the properties of the integral you have learnt in the lecture and do not need to prove them from scratch. For help with this see theorem 6.12 in Rudin.

2.3 Question 2

- Because f is monotone, f takes on in any interval $[x_{i-1}, x_i]$ its maximum and minimum value *in the edges of the interval*.

2.4 Question 4

- Use your result from question 3 to ascertain that

$$\int \frac{Ax+B}{x^2+2a+b}dx = \frac{A}{2}\log\left(\left|x^2+2a+b\right|\right) + \frac{B-aA}{b-a^2}\arctan\left(\frac{x+a}{\sqrt{b-a^2}}\right)$$

- Use fraction decomposition (heavily).

2.5 Question 6

- Write $\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+4}} + \cdots + \frac{1}{\sqrt{2n^2}} = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}}$.
- Define the function $f : [0, 1] \to \mathbb{R}$ by $x \mapsto \frac{1}{\sqrt{1+x^2}}$.
- Define the partition $P_n := \left\{ \frac{k}{n} \mid k \in \{0, \ldots, n\} \right\}$.
- Estimate the sum of the series with the integral (somehow).