Analysis 1
Recitation Session of Week 14

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1 Exercise Sheet Number 12

1.1 Question 1

- For all $(f,\, g) \in \left([a,\, b]^{\mathbb{R}}\right)^2$ define $\langle f,\, g\rangle := \int_a^b f(x)\, g(x)\, dx$.
- *Claim*: $C^0([a,\, b],\, \mathbb{R})$ together with $\langle\cdot,\cdot\rangle$ defined above gives rise to a (real) inner product space.
  Proof:
  - *Claim*: $C^0([a,\, b],\, \mathbb{R})$ is a (real) vector space.
    Proof:
    * Define 'addition' as a map $\left(C^0([a,\, b],\, \mathbb{R})\right)^2 \to C^0([a,\, b],\, \mathbb{R})$ by: $\forall (f,\, g) \in \left[C^0([a,\, b],\, \mathbb{R})\right]^2$, $f+g := (x \mapsto f(x)+g(x)\, \forall x \in [a$
      This map is well defined because of the theorem that says that the sum of two continuous maps is again continuous.
    * We must establish that this 'addition' operation endows $C^0([a,\, b],\, \mathbb{R})$ with the structure of a commutative group:
      · The identity element of the group is given by $(x \mapsto 0 \forall x \in [a,\, b]) \in C^0([a,\, b],\, \mathbb{R})$ because constant maps are continuous.
      · The inverse element of $f \in C^0([a,\, b],\, \mathbb{R})$ is $(x \mapsto -f(x)\, \forall x \in [a,\, b]) \in C^0([a,\, b],\, \mathbb{R})$ because multiplication of a map by -1 leaves a continuous map continuous.
      · Addition is associative due to associativity of addition in $\mathbb{R}$.
      · Addition is commutative due to commutativity of addition in $\mathbb{R}$.
    * Define 'scalar multiplication' as a map $\mathbb{R} \times C^0([a,\, b],\, \mathbb{R}) \to C^0([a,\, b],\, \mathbb{R})$ by $\forall (\alpha,\, f) \in \mathbb{R} \times C^0([a,\, b],\, \mathbb{R})$, $\alpha f := (x \mapsto \alpha f(x)\, \forall x \in [a,\, b])$. This map is well defined because multiplication of a continuous map by a constant is again continuous.
    * We must to establish three properties of the two 'scalar multiplication' and 'addition' maps:
      1. $\forall (\alpha,\, f,\, g) \in \mathbb{R} \times \left[C^0([a,\, b],\, \mathbb{R})\right]^2$, $\alpha(f+g) = \alpha f + \alpha g$ indeed:
         · $\forall x \in [a,\, b]\; \alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x)$ because of distributivity in $\mathbb{R}$.
      2. $\forall (\alpha,\, \beta,\, f) \in \mathbb{R}^2 \times C^0([a,\, b],\, \mathbb{R})$, $(\alpha+\beta) f = \alpha f + \beta f$ and $(\alpha\beta) f = \alpha(\beta f)$.
         · $\forall x \in [a,\, b]\; (\alpha+\beta) f(x) = \alpha f(x) + \beta f(x)$, thanks to distributivity in $\mathbb{R}$.
         · $\forall x \in [a,\, b]\; (\alpha\beta) f(x) = \alpha(\beta f(x))$ due to associtivity of multplication in $\mathbb{R}$.
      3. $\forall f \in C^0([a,\, b],\, \mathbb{R})$, $1f = f$
         · Indeed, as $\forall x \in [a,\, b]\; 1f(x) = f(x)$.
  - Now we need to establish that $\langle\cdot,\cdot\rangle$ is indeed an inner product. It is a map from $\left[C^0([a,\, b],\, \mathbb{R})\right]^2 \to \mathbb{R}$ because the integral produces a real number. It obeys the properties of the inner product. $\forall (f,\, g) \in C^0([a,\, b],\, \mathbb{R})$,
    1. Symmetric:

$$\begin{aligned}\langle f,\, g\rangle &\equiv \int_a^b f(x)\, g(x)\, dx \\ &= \int_a^b g(x)\, f(x)\, dx \\ &\equiv \langle g,\, f\rangle\end{aligned}$$

    2. Positive:

$$\begin{aligned}\langle f,\, f\rangle &\equiv \int_a^b [f(x)]^2\, dx \\ &\leqslant (b-a) \underbrace{\min\left(\left\{ [f(x)]^2 \,\middle|\, x \in [a,\, b] \right\}\right)}_{\geqslant 0}\end{aligned}$$

3. Zero iff zero vector:

$$\begin{aligned}\langle 0, 0\rangle &= \int_a^b 0 dx \\ &= 0\end{aligned}$$

and if $\langle f, f\rangle = 0$ then $\int_a^b [f(x)]^2 dx = 0$. Now suppose f^2 is not identically zero. Then $\exists x_0 \in [a, b]$ such that $[f(x_0)]^2 > 0$. Because f is continuous, f^2 is also continuous. So $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that if $|x - x_0| < \delta(\varepsilon)$ then $\left|[f(x)]^2 - [f(x_0)]^2\right| < \varepsilon$ for all $x \in [a, b]$. Pick $\varepsilon = \frac{1}{2}[f(x_0)]^2 > 0$. Then $[f(x)]^2 > \frac{1}{2}[f(x_0)]^2$ for all $x \in \left[x_0 - \delta\left(\frac{1}{2}[f(x_0)]^2\right), x_0 + \delta\left(\frac{1}{2}[f(x_0)]^2\right)\right]$. Then a lower sum on a partition that contains the interval $\left[x_0 - \delta\left(\frac{1}{2}[f(x_0)]^2\right), x_0 + \delta\left(\frac{1}{2}[f(x_0)]^2\right)\right]$ is larger than or equal to $[f(x_0)]^2 \delta\left(\frac{1}{2}[f(x_0)]^2\right) > 0$. But the lower sums become only *larger* as the partitions become finer (Theorem 6.4 in Rudin). As a result, $\int_a^b [f(x)]^2 dx > 0$, which is a contradiction to the initial hypotheis that $\int_a^b [f(x)]^2 dx = 0$.

4. Linearity in first slot:
Let $(\alpha, \beta, h) \in \mathbb{R}^2 \times \left[C^0([a, b], \mathbb{R})\right]^2$ be given. Then we want to show that

$$\langle \alpha f + \beta g, h\rangle = \alpha \langle f, h\rangle + \beta \langle g, h\rangle$$

which follows easily from Rudin's Theorem 6.12:

$$\begin{aligned}\langle \alpha f + \beta g, h\rangle &\equiv \int_a^b \{[\alpha f(x) + \beta g(x)] h(x)\} dx \\ &= \int_a^b [\alpha f(x) h(x) + \beta g(x) h(x)] dx \\ &= \int_a^b [\alpha f(x) h(x)] dx + \int_a^b [\beta g(x) h(x)] dx \\ &= \alpha \int_a^b [f(x) h(x)] dx + \beta \int_a^b [g(x) h(x)] dx \\ &\equiv \alpha \langle f, h\rangle + \beta \langle g, h\rangle\end{aligned}$$

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2 Holiday Exercise Sheet (Number 13)

2.1 Convex Functions

(question 5.23 in Rudin)

- Let $f \in (a, b)^{\mathbb{R}}$ be given.
- f is called *convex* iff

$$\boxed{f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y)}$$

  for all $(x, y, \lambda) \in (a, b)^2 \times (0, 1)$.
- *Claim*: If f is convex then f is continuous.
  Proof:

  1. *Claim*: $\forall (y, x_0, x) \in (a, b)^3$ such that $a < y < x_0 < x < b$ the following relation holds

  $$\frac{f(x_0) - f(y)}{x_0 - y} \leqslant \frac{f(x) - f(y)}{x - y} \leqslant \frac{f(x) - f(x_0)}{x - x_0}$$

  Proof:
  - Define $\lambda := \frac{x_0 - y}{x - y}$. Note that $\lambda \in (0, 1)$ by definition.
  - Then $1 - \lambda = 1 - \frac{x_0 - y}{x - y} = \frac{x - y - x_0 + y}{x - y} = \frac{x - x_0}{x - y}$ and so

  $$\begin{aligned}(1-\lambda)(x - y) &= x - x_0 \\ x - y - \lambda(x - y) &= x - x_0 \\ -y - \lambda(x - y) &= -x_0 \\ x_0 &= \lambda x + (1-\lambda) y\end{aligned}$$

- Thus we have

$$\begin{aligned} f(x_0) &= f(\lambda x + (1-\lambda) y) \\ &\overset{\text{convexivity}}{\leqslant} \lambda f(x) + (1-\lambda) f(y) \\ &= \frac{x_0 - y}{x - y} f(x) + \frac{x - x_0}{x - y} f(y) \end{aligned}$$

and so

$$\begin{aligned} (x-y) f(x_0) &\leqslant (x_0 - y) f(x) + (x - x_0) f(y) \\ (x-y) f(x_0) - (x-y) f(y) &\leqslant (x_0 - y) f(x) + (x - x_0) f(y) - (x-y) f(y) \\ (x-y)[f(x_0) - f(y)] &\leqslant (x_0 - y)[f(x) - f(y)] \\ \frac{f(x_0) - f(y)}{x_0 - y} &\leqslant \frac{f(x) - f(y)}{x - y} \end{aligned}$$

We also have

$$\begin{aligned} (x-y) f(x_0) &\leqslant (x_0 - y) f(x) + (x - x_0) f(y) \\ -(x-y) f(x_0) &\geqslant -(x_0 - y) f(x) - (x - x_0) f(y) \\ (x-y) f(x) - (x-y) f(x_0) &\geqslant (x-y) f(x) - (x_0 - y) f(x) - (x - x_0) f(y) \\ (x-y)[f(x) - f(x_0)] &\geqslant (x - x_0)[f(x) - f(y)] \\ \frac{f(x) - f(y)}{x - y} &\leqslant \frac{f(x) - f(x_0)}{x - x_0} \end{aligned}$$

2. Let $(y, \alpha, x_0, \beta, x) \in (a, b)^5$ be given such that $y < \alpha < x_0 < \beta < x$. Then by the preceding claim, we have that

$$\frac{f(x_0) - f(y)}{x_0 - y} \leqslant \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leqslant \frac{f(\beta) - f(x_0)}{\beta - x_0} \leqslant \frac{f(x) - f(x_0)}{x - x_0}$$

3. Define $\begin{cases} m & := \frac{f(x_0) - f(y)}{x_0 - y} \\ M & := \frac{f(x) - f(x_0)}{x - x_0} \end{cases}$
4. Thus we have

$$m \leqslant \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leqslant \frac{f(\beta) - f(x_0)}{\beta - x_0} \leqslant M$$

   or

$$\begin{cases} m \leqslant \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leqslant M \\ m \leqslant \frac{f(\beta) - f(x_0)}{\beta - x_0} \leqslant M \end{cases}$$

5. For the first inequality, $m \leqslant \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leqslant M$, or

$$m(x_0 - \alpha) \leqslant [f(x_0) - f(\alpha)] \leqslant M(x_0 - \alpha)$$

- If $m > 0$ and $M > 0$, define $\delta := \frac{\varepsilon}{M}$.
  * Then if $0 < x_0 - \alpha < \delta$ then $[f(x_0) - f(\alpha)] < \varepsilon$.
  * Because $(x_0 - \alpha) m > 0$, $(x_0 - \alpha) m > -\varepsilon$ so that $f(x_0) - f(\alpha) > -\varepsilon$, so that $|f(x_0) - f(\alpha)| < \varepsilon$.
  * Thus

$$\lim_{\alpha \to x_0^-} f(\alpha) = f(x_0)$$

- If $m < 0$ and $M > 0$, define $\delta := \varepsilon \min\left(\left\{ \frac{1}{|m|}, \frac{1}{M} \right\}\right)$.
  * Then the right hand side is fulfilled.
  * The left hand side has:
    · $x_0 - \alpha < \delta$ then $m(x_0 - \alpha) > -\varepsilon$
    · thus $f(x_0) - f(\alpha) > -\varepsilon$
  * Thus $|f(x_0) - f(\alpha)| < \varepsilon$.
- If $m < 0$ and $M < 0$, define $\delta := \varepsilon \frac{1}{|m|}$.
  * Then $f(x_0) - f(\alpha) < 0 < \varepsilon$ and $f(x_0) - f(\alpha) > (x_0 - \alpha) m > -\varepsilon$
- If $m = 0$ and so $M \geqslant 0$, define $\delta := \frac{\varepsilon}{M}$ (unless $M = 0$, in which case any δ will do).
- If $M = 0$ and so $m \leqslant 0$, define $\delta := \frac{\varepsilon}{|m|}$ unless $m = 0$ and then any δ will do.

6. Final conclusion:
$$\lim_{\alpha\to x_0^-} f(\alpha) = f(x_0)$$
7. In a similar analysis we can conclude also that
$$\lim_{\beta\to x_0^+} f(\beta) = f(x_0)$$
8. That means that $\lim_{x\to x_0} f(x) = f(x_0)$. Thus according to Rudin's Theorem 4.6 f is continuous at x_0.
9. As x_0 was arbitrary, f is continuous.

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- *Claim*: Every increasing convex function of a convex function is convex.
  Proof:

1. Let $(f, g) \in \left[(a, b)^{\mathbb{R}}\right]^2$ be given such that f is convex and g is convex and increasing.
2. Define $h \in (a, b)^{\mathbb{R}}$ by $h := g \circ f$.
3. The statement of the claim is then that h is convex itself, as well.
4. That means that h should obey
$$\boxed{h(\lambda x + (1-\lambda) y) \leqslant \lambda h(x) + (1-\lambda) h(y)}$$
for all $(x, y, \lambda) \in (a, b)^2 \times (0, 1)$.
5. $h(\lambda x + (1-\lambda) y) = g(f(\lambda x + (1-\lambda) y))$ by definition.
6. Use the fact that f is convex, so that $f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y)$.
7. Use the fact that g is increasing, so if $\alpha \leqslant \beta$ then $g(\alpha) \leqslant g(\beta)$. In our case, that means that $g(f(\lambda x + (1-\lambda) y)) \leqslant g(\lambda f(x) + (1-\lambda) f(y))$.
8. Use the fact that g is convex so that $g(\lambda f(x) + (1-\lambda) f(y)) \leqslant \lambda g(f(x)) + (1-\lambda) g(f(y))$.

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2.2 Differential Equations

2.3 Fourier Series

2.4 Riemann-Stieltjes Integral

2.5 Convolution and the Stone Weierstrass Theorem