1 Exercise Sheet Number 12

1.1 Question 1

• For all \((f, g) \in ([a, b]^R)^2\) define \(\langle f, g \rangle \equiv \int_a^b f(x) g(x) \, dx\).

• Claim: \(C^0([a, b], R)\) together with \(\langle \cdot , \cdot \rangle\) defined above gives rise to a (real) inner product space.

  Proof:

  – Claim: \(C^0([a, b], R)\) is a (real) vector space.

    Proof:

    * Define ’addition’ as a map \((C^0([a, b], R))^2 \to C^0([a, b], R))\) by: \(\forall (f, g) \in (C^0([a, b], R))^2\), \(f \circ g := (x \mapsto f(x) + g(x)) \forall x \in [a, b]\). This map is well defined because of the theorem that says that the sum of two continuous maps is again continuous.

    * We must establish that this ’addition’ operation endows \(C^0([a, b], R)\) with the structure of a commutative group:

      - The identity element of the group is given by \((x \mapsto 0)\) \forall x \in [a, b] \) because \(\) constant maps are continuous.

      - The inverse element of \(f \in C^0([a, b], R)\) is \((x \mapsto -f(x)) \forall x \in [a, b]\) because multiplication of a map by \(-1\) leaves a continuous map continuous.

      - Addition is associative due to associativity of addition in \(R\).

      - Addition is commutative due to commutativity of addition in \(R\).

    * Define ’scalar multiplication’ as a map \(R \times C^0([a, b], R) \to C^0([a, b], R)\) by \(\forall (\alpha, f) \in R \times C^0([a, b], R)\), \(\alpha f := (x \mapsto \alpha f(x)) \forall x \in [a, b]\). This map is well defined because multiplication of a continuous map by a constant is again continuous.

    * We must to establish three properties of the two ’scalar multiplication’ and ’addition’ maps:

      1. \(\forall (\alpha, f, g) \in R \times (C^0([a, b], R))^2\), \(\alpha (f + g) = \alpha f + \alpha g\) indeed:

         - \(\forall x \in [a, b]\) \(\alpha (f(x) + g(x)) = \alpha f(x) + \alpha g(x)\) because of distributivity in \(R\).

      2. \(\forall (\alpha, \beta, f) \in R \times C^0([a, b], R)\), \((\alpha + \beta) f = \alpha f + \beta f\) and \((\alpha \beta) f = \alpha (\beta f)\).

         - \(\forall x \in [a, b]\) \((\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)\), thanks to distributivity in \(R\).

         - \(\forall x \in [a, b]\) \((\alpha \beta) f(x) = \alpha (\beta f(x))\) due to associativity of multiplication in \(R\).

      3. \(\forall f \in C^0([a, b], R)\), \(1 f = f\)

         - Indeed, as \(\forall x \in [a, b]\) \(1 f(x) = f(x)\).

    – Now we need to establish that \(\langle \cdot , \cdot \rangle\) is indeed an inner product. It is a map from \([C^0([a, b], R)]^2 \to R\) because the integral produces a real number. It obeys the properties of the inner product. \(\forall (f, g) \in C^0([a, b], R)\),

      1. Symmetric:

         \[
         \langle f, g \rangle \equiv \int_a^b f(x) g(x) \, dx
         = \int_a^b g(x) f(x) \, dx
         \equiv \langle g, f \rangle
         \]

      2. Positive:

         \[
         \langle f, f \rangle \equiv \int_a^b [f(x)]^2 \, dx
         \leq (b - a) \min \left( \left\{ \frac{|f(x)|^2}{x \in [a, b]} \right\} \right)_{|f| \geq 0}
         \]


Analysis 1
Recitation Session of Week 14

Jacob Shapiro
December 19, 2014
3. Zero iff zero vector:

\[ \langle 0, 0 \rangle = \int_a^b 0 \, dx = 0 \]

and if \((f, f) = 0\) then \(\int_a^b |f(x)|^2 \, dx = 0\). Now suppose \(f^2\) is not identically zero. Then \(\exists x_0 \in [a, b]\) such that \(|f(x_0)|^2 > 0\). Because \(f\) is continuous, \(f^2\) is also continuous. So \(\forall \varepsilon > 0 \exists \delta (\varepsilon) > 0\) such that if \(|x - x_0| < \delta (\varepsilon)\) then \(|f(x)| > |f(x_0)|\) for all \(x \in (x_0 - \delta, x_0 + \delta)\). Pick \(\varepsilon = \frac{1}{2} |f(x_0)|^2 > 0\). Then \(|f(x)|^2 > \frac{1}{2} |f(x_0)|^2\) for all \(x \in (x_0 - \delta, x_0 + \delta)\). Then a lower sum on a partition that contains the interval \([x_0 - \delta, x_0 + \delta]\) is larger than or equal to \(\frac{1}{2} |f(x_0)|^2 \delta > 0\). But the lower sums become only larger as the partitions become finer (Theorem 6.4 in Rudin). As a result, \(\int_a^b |f(x)|^2 \, dx > 0\), which is a contradiction to the initial hypothesis that \(\int_a^b |f(x)|^2 \, dx = 0\).

4. Linearity in first slot:
Let \((\alpha, \beta, h) \in \mathbb{R}^2 \times \left[ C^0 ([a, b], \mathbb{R}) \right]^2\) be given. Then we want to show that

\[ \langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle \]

which follows easily from Rudin’s Theorem 6.12:

\[
\langle \alpha f + \beta g, h \rangle = \int_a^b ([\alpha f (x) + \beta g (x)] h (x)) \, dx
= \int_a^b [\alpha f (x) h (x) + \beta g (x) h (x)] \, dx
= \int_a^b [\alpha f (x) h (x)] \, dx + \int_a^b [\beta g (x) h (x)] \, dx
= \alpha \int_a^b [f (x) h (x)] \, dx + \beta \int_a^b [g (x) h (x)] \, dx
= \alpha \langle f, h \rangle + \beta \langle g, h \rangle
\]

\[\blacksquare\]

## 2 Holiday Exercise Sheet (Number 13)

### 2.1 Convex Functions

(question 5.23 in Rudin)

- Let \(f \in (a, b)\mathbb{R}\) be given.
- \(f\) is called convex iff

\[ f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \]

for all \((x, y, \lambda) \in (a, b)^2 \times (0, 1)\).

- **Claim:** If \(f\) is convex then \(f\) is continuous.

**Proof:**

1. **Claim:** \(\forall (y, x_0, x) \in (a, b)^3\) such that \(a < y < x_0 < x < b\) the following relation holds

\[
\frac{f(x_0) - f(y)}{x_0 - y} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(x_0)}{x - x_0}
\]

**Proof:**

- Define \(\lambda := \frac{x_0 - y}{x - y}\). Note that \(\lambda \in (0, 1)\) by definition.

- Then \(1 - \lambda = 1 - \frac{x_0 - y}{x - y} = \frac{x - y - x_0 + y}{x - y} = \frac{x - x_0}{x - y}\) and so

\[
(1 - \lambda)(x - y) = x - x_0
\]

\[
x - y - \lambda(x - y) = x - x_0
\]

\[
-x - \lambda(x - y) = -x_0
\]

\[
x_0 = \lambda x + (1 - \lambda) y
\]
Thus we have

\[ f(x_0) = f(\lambda x + (1 - \lambda) y) \]

Convexity

\[ M f(x) + (1 - \lambda) f(y) \]

\[ x_0 - \frac{y}{x - y} f(x) + \frac{x - x_0}{x - y} f(y) \]

and so

\[ (x - y) f(x_0) \leq (x_0 - y) f(x) + (x - x_0) f(y) \]

\[ (x - y) f(x_0) - (x - y) f(y) \leq (x_0 - y) f(x) + (x - x_0) f(y) - (x - y) f(y) \]

\[ (x - y)(f(x_0) - f(y)) \leq (x_0 - y)(f(x) - f(y)) \]

\[ \frac{f(x_0) - f(y)}{x_0 - y} \leq \frac{f(x) - f(y)}{x - y} \]

We also have

\[ (x - y) f(x_0) \leq (x_0 - y) f(x) + (x - x_0) f(y) \]

\[ -(x - y) f(x_0) \geq -(x_0 - y) f(x) - (x - x_0) f(y) \]

\[ (x - y)[f(x) - f(x_0)] \geq (x_0 - y)[f(x) - f(y)] \]

\[ \frac{f(x) - f(x_0)}{x - y} \leq \frac{f(x) - f(y)}{x - x_0} \]

2. Let \((y, \alpha, x_0, \beta, x) \in (a, b)^5\) be given such that \(-\alpha < x_0 < \beta < x\). Then by the preceding claim, we have that

\[ \frac{f(x_0) - f(y)}{x_0 - y} \leq \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leq \frac{f(\beta) - f(x_0)}{\beta - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0} \]

3. Define

\[ m := \frac{f(x_0) - f(y)}{x_0 - y}, \quad M := \frac{f(x) - f(x_0)}{x - x_0} \]

4. Thus we have

\[ m \leq \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leq \frac{f(y) - f(x_0)}{\beta - x_0} \leq M \]

or

\[ \left\{ \begin{array}{l}
m \leq \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \\
m \leq \frac{f(y) - f(x_0)}{\beta - x_0}
\end{array} \right\} \leq M \]

5. For the first inequality, \(m \leq \frac{f(x_0) - f(\alpha)}{x_0 - \alpha} \leq M\), or

\[ m(x_0 - \alpha) \leq [f(x_0) - f(\alpha)] \leq M(x_0 - \alpha) \]

- If \(m > 0\) and \(M > 0\), define \(\delta := \frac{1}{M} \).
  * Then if \(0 < x_0 - \alpha < \delta\) then \(|f(x_0) - f(\alpha)| \leq \varepsilon\).
  * Because \((x_0 - \alpha) m > 0\), \((x_0 - \alpha) m > -\varepsilon\) so that \(f(x_0) - f(\alpha) > -\varepsilon\), so that \(|f(x_0) - f(\alpha)| < \varepsilon\).
  * Thus

\[ \lim_{\alpha \to x_0} f(\alpha) = f(x_0) \]

- If \(m < 0\) and \(M > 0\), define \(\delta := \varepsilon \min \left( \frac{1}{|m|}, \frac{1}{M} \right) \).
  * Then the right hand side is fulfilled.
  * The left hand side has:
    * \(x_0 - \alpha < \delta\) then \(m(x_0 - \alpha) > -\varepsilon\)
    * Thus \(f(x_0) - f(\alpha) > -\varepsilon\)
  * Thus \(|f(x_0) - f(\alpha)| < \varepsilon\).
- If \(m < 0\) and \(M < 0\), define \(\delta := \varepsilon \frac{1}{|m|}\).
  * Then \(f(x_0) - f(\alpha) < 0 < \varepsilon\) and \(f(x_0) - f(\alpha) > (x_0 - \alpha) m > -\varepsilon\)
- If \(m = 0\) and \(M > 0\), define \(\delta := \frac{\varepsilon}{M}\) (unless \(M = 0\), in which case any \(\delta\) will do).
- If \(M = 0\) and \(m \leq 0\), define \(\delta := \frac{\varepsilon}{|m|}\) unless \(m = 0\) and then any \(\delta\) will do.
6. Final conclusion: \( \lim_{\alpha \to x_0^-} f(\alpha) = f(x_0) \)

7. In a similar analysis we can conclude also that \( \lim_{\beta \to x_0^+} f(\beta) = f(x_0) \)

8. That means that \( \lim_{x \to x_0} f(x) = f(x_0) \). Thus according to Rudin’s Theorem 4.6 \( f \) is continuous at \( x_0 \).

9. As \( x_0 \) was arbitrary, \( f \) is continuous.

Claim: Every increasing convex function of a convex function is convex.

Proof:

1. Let \((f, g) \in ([a, b]^R)^2 \) be given such that \( f \) is convex and \( g \) is convex and increasing.

2. Define \( h \in (a, b)^R \) by \( h := g \circ f \).

3. The statement of the claim is then that \( h \) is convex itself, as well.

4. That means that \( h \) should obey

   \[
   h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda) h(y)
   \]

   for all \((x, y, \lambda) \in (a, b)^2 \times (0, 1)\).

5. \( h(\lambda x + (1 - \lambda)y) = g(f(\lambda x + (1 - \lambda)y)) \) by definition.

6. Use the fact that \( f \) is convex, so that \( f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) \).

7. Use the fact that \( g \) is increasing, so if \( \alpha \leq \beta \) then \( g(\alpha) \leq g(\beta) \). In our case, that means that \( g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda) f(y)) \).

8. Use the fact that \( g \) is convex so that \( g(\lambda f(x) + (1 - \lambda) f(y)) \leq \lambda g(f(x)) + (1 - \lambda) g(f(y)) \).

2.2 Differential Equations

2.3 Fourier Series

2.4 Riemann-Stieltjes Integral

2.5 Convolution and the Stone Weierstrass Theorem