Analysis 1 - Recitation of Week 2

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1 Tips for Exercise Sheet Number 2

1.1 Subsets of $\mathbb{R}$

- Up until now we did not restrict the kind of elements in our sets. However, in this exercise we will mainly be concerned with subsets of $\mathbb{R}$, that is, sets whose elements are real numbers. These sets can be finite: $\{\, 1, 2, \pi \,\}$ or they can be infinite: $\{\, x \in \mathbb{R} \mid x > 0 \,\}$ or $\left\{x \in \mathbb{R} | x^2 < 2\right\}$.
- These sets have order on them, which they inherit from $\mathbb{R}$. To have an order, means, in general:
  1. $\forall x \in A, \forall y \in A$: $x < y \veebar x = y \veebar y < x$ (Observe new notation for XOR operation: $\veebar$ means either option one or option two, but NOT the both together).
  2. $\forall x \in A, \forall y \in A, \forall z \in A$, if $x < y$ and $y < z$ then $x < z$.

1.2 Bounded Sets

Let two sets, A_1 and A_2 be given, such that $A_1 \subset A_2$ and A_2 is an ordered set.

- Definition: A_1 is called "bounded from above" iff $\exists M \in A_2$ such that $\forall x \in A_1$, $x \leq M$. In this case, M is called "an upper bound on A_1".
- Definition: A_1 is called "bounded from below" iff $\exists m \in A_2$ such that $\forall x \in A_1$, $x \geq m$. In this case, m is called "a lower bound on A_1".
- Example of sets bounded from above:
  - $\underbrace{\{\, 1, 2, 3 \,\}}_{A_1} \subset \underbrace{\mathbb{R}}_{A_2}$ is bounded above with upper bounds: 3, 3.1, 4, 5, 100 and so on (infinitely man)
  - $\underbrace{\left\{\, x \in \mathbb{R} \,\middle|\, x^2 < 2 \,\right\}}_{A_1} \subset \underbrace{\mathbb{R}}_{A_2}$ is bounded above with upper bounds: $\sqrt{2} \approx 1.41421$, 1.5, 2, etc.
    Note how the an upper bound is of course *not* required to belong to the given set.

- Examples of sets *not* bounded from above:
  - $\mathbb{R} \subset \mathbb{R}$ is not bounded from above: there is no element of $\mathbb{R}$ that we can find which will be larger than any imaginable real number.
  - $\mathbb{N} \subset \mathbb{R}$ is not bounded from above for the same reason (but it is bounded from below, by 0 and everything less than that).
  - $2\mathbb{N} \equiv \{0, 2, 4, \dots\}$ is also not bounded from above (but from below it is).
- Examples of sets not bounded from above *or* below:
  - $\mathbb{R} \subset \mathbb{R}$
  - $\mathbb{Z} \subset \mathbb{R}$ (note that $\mathbb{N}$ is bounded from below)

1.3 Supremum and Infimum

Let two sets, A_1 and A_2 be given, such that $A_1 \subset A_2$ and A_2 is an ordered set.

- Definition: A "supremum" on a set A_1 which is bounded above is defined (if it exists) as an upper bound on A_1 which is also the smallest possible upper bound.
  - Note there are cases when this doesn't exist, as we shall see (just as there are sets with no smallest number).
  - Also called "least upper bound".
  - In symbols, $\sup(A_1)$
  - For a supremum to exist we have (now in symbols) these two conditions:
    * $\forall x \in A_1 : x \leq \sup(A_1)$ ($\sup(A_1)$ is an upper bound)
    * $\nexists \alpha \in A_2 : \alpha < \sup(A_1)$ and such that $\forall x \in A_1,\ x \leq \alpha$ (no smaller upper bound).
  - Examples:
    * The most stupid examples are when our set has a maximum, and then the supremum is just this maximum: $\sup(\{1, 2, 3\}) = \max(\{1, 2, 3\}) = 3$ (there is always a maximum for finite sets) or $\sup([0, 5]) = \max([0, 5]) = 5$.
    * $A_1 := \left\{ -\frac{1}{n} \,\middle|\, n \in \mathbb{N}\backslash\{0\} \right\} \equiv \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\right\} \subset \mathbb{R}$ is a given set. It is clearly bounded above (by any positive number as well as by zero). Claim: $\sup(A_1) = 0$. To prove this statement we must show that the two conditions hold. The first, which is, that 0 is an upper bound on A_1 is clearly true: since all the numbers in our set our negative, any non-negative number, includign zero, will do the job. The second condition means there is no smaller lower bound. Intuitively, this has to do with

the fact that our elements in the set are getting closer and closer towards 0 (though never reaching it). To be more precise, assume otherwise that there *is* a smaller upper bound, α, on A_1. That means, by assumption, that $\alpha < 0$. But for any negative number, however close to zero, we can always find a large enough $n \in \mathbb{N}$ so that $-\frac{1}{n} > \alpha$ and so α is no longer an upper bound on A_1!

  * $\underbrace{\{ x \in \mathbb{Q} \mid x^2 < 2 \}}_{A_1} \subset \underbrace{\mathbb{R}}_{A_2}$. This set is clearly bounded above by $\sqrt{2}$ (as well as anything above that). Our claim is that $\sup(A_1) = \sqrt{2}$ (even though, since $\sqrt{2} \notin \mathbb{Q}$, $\sup(A_1) \notin A_1$. To verify the second condition, assume we have found a smaller upper bound α. This means that $\alpha < \sqrt{2}$. Thus, $\alpha + \beta = \sqrt{2}$, for *some* $\beta \in \mathbb{R}$. So between α and $\alpha + \frac{1}{2}\beta$ (two arbitrary real numbers) we can always find a rational (this is because the rationals are dense in the reals, a fact we will see later) in between them, call it q: $\alpha < q < \alpha + \frac{1}{2}\beta < \sqrt{2}$. So that $q \in A_1$ and $q > \alpha$ (that is, α is not an upper bound on A_1).

- Approximation property: We can convert the second property of the supremum definition into an "approximation property" of sorts:
  * $\nexists \alpha \in A_2 : \alpha < \sup(A_1)$ and such that $\forall x \in A_1$, $x \leq \alpha$ is equivalent to
  * $\forall \alpha \in A_2$: $\alpha < \sup(A_1)$, $\exists x_\alpha \in A_1$ such that $x_\alpha > \alpha$.
  * But to pick such an α is not hard at all: all it has to obey is that it's smaller than the supremum. So we can start with $\sup(A_1)$ and push it down just a tiny bit. How much, by some small positive number $\varepsilon > 0$. So $\alpha := \sup(A_1) - \varepsilon$.
  * Thus $\forall \varepsilon > 0 \ \exists x_\varepsilon \in A_1$ such that $x_\varepsilon > \sup(A_1) - \varepsilon$. But $\sup(A_1)$ is also the upper bound, so we can combine these two facts togethre to say that:
  * $\boxed{\forall \varepsilon > 0 \ \exists x_\varepsilon \in A_1 \text{ such that } \sup(A_1) - \varepsilon < x_\varepsilon \leq \sup(A_1).}$
  * This "approximation property" is extremely important in proofs, especially when the supremum is taken over some crazy set and we have no hope to actually calculate what the supremum is. So remember it!

• Definition: An "infimum" on a set E_1 which is bounded below is defined (if it exists) as a lower bound on A_1 which is also the largest possible lower bound.

  - Again, this might not exist.
  - Also called "greatest lower bound".
  - If minimum exists, then it is just the minimum (always for finite sets).

- Examples:
  - $A_1 := \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \setminus \{ 0 \} \right\} \equiv \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \subset \mathbb{R}$ is a given set. As before, $\inf (A_1) = 0$.
  - Note that this exhibits a general phenomenon (part of the homework): for a given set A_1 which is bounded above and has $\sup (A_1)$, $-A$ is bounded below and has $\inf (-A_1)$ which is equal to $\inf (-A_1) = -\sup (A_1)$. By the way, $-A \equiv \{ -a \mid a \in A \}$.
  - To solve question 1 on the homework, "follow your nose": prove that $-\sup (A)$ is a lower bound on $-A$ and then go on to prove it is the largest lower bound (to do this, assume otherwise, that is, that there is a larger lower bound, then somehow try to "transfer" this larger lower bound on $-A$ into a lower upper bound on A and reach a contradiction). You will need to use the fact that if $\alpha < \beta$ then $-\alpha > -\beta$ and vice versa, as well as $\alpha \leq \beta$ implies $-\alpha \geq -\beta$. For the second part you can follow a very similar procedure, only note that $x \leq M$ implies $\frac{1}{M} \leq \frac{1}{x}$.

1.4 Injectivity, Surjectivity and Bijectivity

1.4.1 Injectivity

- Definition: given two sets A and B and a function between them $f : A \to B$, f is called "an injective function" or "an injection" iff $\forall a_1 \in A, \forall a_2 \in A$, $f (a_1) = f (a_2)$ implies $a_1 = a_2$.
- In words: two different elements in the source set cannot go to the SAME element in the destination set: $\forall a_1 \in A$, $\forall a_2 \in A$, $a_1 \neq a_2$ implies $f (a_1) \neq f (a_2)$.
- Do not confuse with the general condition on a rule between two sets to be a mathematical function: two different elements in the destination set cannot have come from the SAME element in the source set: $f (a)$ has only one value for each $a \in A$.
- Which on of the conditions to use depends on the problem at hand.
- Example: $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ is *not* injective because 2 and -2 (two different elements) go to the same element in the destination, namely, 4.
- Example: $f : [0, \infty) \to \mathbb{R}$ given by $x \mapsto x^2$ *is* injective, because now we have exluded the negative numbers in the source set, and as a result, we recover the condition for injectivity.
- Example: $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^3$ is injective.
- Question: is it possible to have an injective function if the source set is "larger" (for finite sets) than the destination set? (pigeon hole principle).

- Define inverse image as $f^{-1}(E) := \{ a \in A \mid f(a) \in E \}$ for all $E \subset B$ (draw picture).

- Another characterization: f is injective iff $\left|f^{-1}(\{x\})\right| = 1 \forall x \in B$.

- Note: if f is injective then we can unambiguously "return" from B back to A and so we can define an inverse function f^{-1}.

1.4.2 Surjectivity

- Definition: given two sets A and B and a function between them $f : A \to B$, f is called "an surjective function" or "a surjection" iff $\forall b \in B \exists a \in A$ such that $f(a) = b$.

- Note that in this definition a is absolutely not required to be unique! Question: when *will* it be unique? (when f is injective).

- Surjectivity means we "cover" the whole of B with f and there is no element of B left unreferenced by f.

- Example: $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ is not surjective: we will never produce a negative number with this map. As we have seen it is not injective.

- Example: $f : [0, \infty) \to [0, \infty)$ given by $x \mapsto x^2$ is surjective (it covers the whole of its destination) and as we've seen it's also injective.

- Question: how to get a surjective function out of a non-surjective function? (redefine the destination set to include only those points which are "referenced" by f)

- Example: $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x$ is surjective. Is it injective?

- Example (less trivial): $f : [0, 2] \to [0, 1]$ given by $x \mapsto x$ if $x \leq 1$ and $x \mapsto x - 1$ otherwise. This function is surjective, but not injective!

- Question: Is it possible to have a surjection if the source set is "smaller" than the destination set?

1.4.3 Bijections

- A bijective function, or a bijection, is a function that is both injective AND surjective. It allows us to go back and forth between the source and destination sets and always reach the same result.

- Example: $f : \{Students\} \to \{Legi\#\}$ is a bijection (has to be!)

- Example: When we have two sets of the same (finite) size, we can always find a bijection between them. (Just draw arrows until all elements have an arrow next to them)

- Example: $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x$ is a bijection. This will always be true: the identity function (the function that does nothing) is always a bijection.

- Example: $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto 2x + 1$ is a bijection.

1.5 Cardinality (Mächtigkeit)

- The cardinality of a set is a measure of the "number of elements of the set".

- Note that in order to measure something (like length or time) we always need a reference object to compare our measure to. For example, the meter (a metal rod kept in Paris) or the time it takes an atom to perfrom 5000 oscillations is one second. The same is true when we want to measure sizes of sets: we need some reference sets to tell us what are the standard "units" of size for sets.

- The actual measuring of the size is performed by putting our object next to the meter rod and seeing if it's bigger, smaller, or equal.

- With sets, we define two sets to have equal "size" if there is a bijection between them. This makes intuitive sense because a bijection between the two sets is a one-to-one correspondence between the two sets.

- For sets of finite size this is very easy: we can always find a bijection between two finite sets if they have the same number of elements.

- For infinite sets we really need to hold on very carefully to the existence of bijections. The pioneer in this field was Georg Cantor who decided that two (infinite) sets have the same cardinality (size) iff there exists a bijection between them. He then went on to identify the "standard units of cardinality" for sets: finite sets, $\mathbb{N}$, $\mathbb{R}$ and so on.

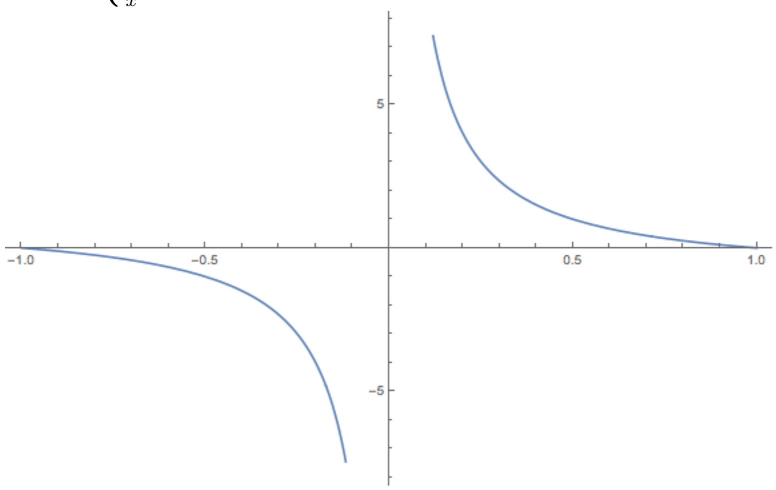
- A set with the same cardinality as $\mathbb{N}$ (a set that has a bijection with $\mathbb{N}$) is called "countable" (abzählbar).

- A set with the same cardinality as $\mathbb{R}$ (a set that has a bijection with $\mathbb{R}$) is called "uncountable" (unzählbar).

- Example: $2\mathbb{N} := \{0, 2, 4, 6, \dots\}$ is countable. To show this, we need to find a bijection with $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. This sound crazy, because intuitively, we *see* that there are twice as many elements in $\mathbb{N}$. But indeed, it is the infinite nature of the sets that defies our intutition. The bijection that exists between the sets is $f : 2\mathbb{N} \to \mathbb{N}$ given by $n \mapsto \frac{1}{2}n$. It is left as homework to verify that this is indeed surjective and injective.

- Other examples: $\mathbb{Z}$ and $\mathbb{Q}$ are also countable. The proof for $\mathbb{Q}$ is not so simple.

- Example: $(-1,\,1)$ has the same cardinality as $\mathbb{R}$: Define $f:(-1,\,1)\to\mathbb{R}$ by $x\mapsto\begin{cases}\frac{1}{x}-1 & \text{if } x>0\\ 0 & \text{if } x=0\\ \frac{1}{x}+1 & \text{if } x<0\end{cases}$



(clearly surjective and injective from the graph)

- Using the above example we can find that any open interval in $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ (how?), Thus, for example, $(0.1,\,0.9)$ has the same cardinality of $\mathbb{R}$.

- What about $(0.1,\,0.9)\cup\mathbb{N}$? Very similar to the homework. We know $(0.1,\,0.9)$ has the cardinality of $\mathbb{R}$, so there is a map $f:(0.1,\,0.9)\to\mathbb{R}$ which is bijective. So define a map $g:(0.1,\,0.9)\cup\mathbb{N}\to\mathbb{R}$ by the following: $g(x):=\begin{cases}2x+1 & \text{if } x\in\mathbb{N}\\ 2f(x) & \text{if } x\notin\mathbb{N} \text{ and } f(x)\in\mathbb{N}\\ f(x) & \text{if } x\notin\mathbb{N} \text{ and } f(x)\in\mathbb{R}\backslash\mathbb{N}\end{cases}$. Think at home why this works.

1.6 Induction with Two Steps

- Sometimes it will be useful to assume two steps below the induction instead of just one. This is usually the case when trying to prove some recursion relation which involves two steps, as in question 4 on the homework.

- Let us construct this new variant of induction based on the old one:

- Suppose we are given a statement $P(k)$ for each $k\in\mathbb{N}$ and we want to prove that this statement holds for all $k\in\mathbb{N}$.

- Define a new statement, $Q(k):=P(k)\wedge P(k+1)$.

- The principle of mathematical induction (now on $Q(k)$) tells us that $(Q(0) \wedge (\forall k \in \mathbb{N}(Q(k) \implies Q(k+1)))) \implies Q(k)\, \forall k \in \mathbb{N}$.
- Translating this to conditions on $P(k)$ we get:
  - $Q(0) \iff P(0) \wedge P(1)$
  - $Q(k) \iff P(k) \wedge P(k+1)$
  - $Q(k+1) \iff P(k+1) \wedge P(k+2)$
- So to make the induction variation we need:
  $((P(0) \wedge P(1)) \wedge (\forall k \in \mathbb{N}((P(k) \wedge P(k+1)) \implies (P(k+1) \wedge P(k+2))))) \implies (P(k) \wedge P(k+1))\, \forall k \in \mathbb{N}$
- Or in a less redundant form:
  $((P(0) \wedge P(1)) \wedge (\forall k \in \mathbb{N}((P(k) \wedge P(k+1)) \implies (P(k+2))))) \implies P(k)\, \forall k \in \mathbb{N}$.

1.7 Question Specific Tips

- For question 4 (b), try to isolate in $\frac{a_{n+1}}{a_n}$ a factor of ϕ and something else. This something else will be a complicated expression. To simplify it, define $\alpha := \frac{1+\sqrt{5}}{1-\sqrt{5}}$ and get an expression of the form
  $\frac{a_{n+1}}{a_n} = \phi \times$ something that depends on α and n.
- Then $\left|\frac{a_{n+1}}{a_n} - \phi\right| = \phi\, |\text{something} - 1|$.
- Try to estimate $|\text{something} - 1|$ by something slightly larger, using the fact that $\frac{1}{|x^n - 1|} \le \frac{1}{|x|^n - 1}$ for our $x < -1$.
- Then pick some n large enough so that $\frac{C}{|x|^n - 1} < \varepsilon$ for some initially selected small ε.
- For question 5:
  - (a): I sent last week.
  - (b): Just plugin the formula recursively (not using (a)).
  - (c): Look at the pattern you found in (b), make an intuitive guess, and prove it using induction and (a).
- For question 6:
  - (a): Don't use induction, rather, find some $m \in \mathbb{N}$ so that $2^m > n$, and try to build an inequality that will eventually lead to a geometric series which will be equal to 2.
  - (b): Write two things that $\sum_{k=0}^{n} \sum_{l=0}^{n} a_k a_l$ is equal to.
  - (c): If two polynomials are the same, then the coefficients are the same order by order. Compare the coefficients of x^n.