Analysis 1
Recitation Session of Week 4

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October 10, 2014

1 Exercise Sheet Number 4

1.1 Complex Numbers

1.1.1 Solutions to Equations

- *Claim:* If $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ is an equation where $a_i \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ solves the equation then so does $\overline{\alpha}$.
  Proof:
  - If $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ holds then so does

$$\begin{aligned} \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} &= \overline{0} \\ \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} &= 0 \\ a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \cdots + a_1 \overline{z} + \overline{a_0} &= 0 \\ a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \cdots + a_1 \overline{z} + \overline{a_0} &= 0 \end{aligned}$$

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- As a result, given one solution of an equation with real coefficients, then the complex conjugate of that solution is yet another solution (if the first solution is not real, otherwise we have the same thing of course as the complex conjugate of a real number is the same number again).
- Another way to find solutions to an equation is to make polynomial division by known solutions, which then simplies the given equation.
- For example, assume $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ has a solution $\alpha \in \mathbb{C}$.
- That means that $(z - \alpha)$ somehow can be factored out of this equation: otherwise it would not be a solution!
- Divide $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ by $(z - \alpha)$ to get $b_{n-1} z^{n-1} + \cdots + b_1 z + b_0$.

- Sometimes $b_{n-1}z^{n-1} + \cdots + b_1 z + b_0 = 0$ is easier to solve. Especially if now $n-1 = 2$.
- Example:
  - $x^3 + x^2 + x + 1 = 0$ has a solution: $x = -1$. (Check by plugging in)
  - So try to factor $(x+1)$ out of it: $x^3 + x^2 + x + 1 = (x+1)\left(x^2+1\right)$ by guessing wisely.
  - But we know how to solve $x^2 + 1 = 0$! So we can now write down all solutions: $x = -1$, $x = \pm i$.

1.1.2 Analytic Geometry

- *Claim:* The area of a rectangle with corners at z_1 and z_2 is given by $|\Re(z_2)\Im(z_2) - \Im(z_1 z_2) + \Re(z_1)\Im(z_1)|$.
  Proof:
  - Define $x_i := \Re(z_i)\,\forall i \in \{1,\,2\}$, $y_i := \Im(z_i)\,\forall i \in \{1,\,2\}$.
  - Then the length of the edges of the rectangle are $|x_2 - x_1|$ and $|y_2 - y_1|$.
  - Thus

    $$\begin{aligned} A &= |x_2 - x_1| \times |y_2 - y_1| \\ &= |\Re(z_2) - \Re(z_1)|\,|\Im(z_2) - \Im(z_1)| \\ &= |\Re(z_2)\Im(z_2) - \Re(z_2)\Im(z_1) - \Re(z_1)\Im(z_2) + \Re(z_1)\Im(z_1)| \\ &= |\Re(z_2)\Im(z_2) - \Im(z_1 z_2) + \Re(z_1)\Im(z_1)| \end{aligned}$$

    because $\Im(z_1 z_2) = \Im((x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)) = x_1 y_2 + x_2 y_1$.

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- *Example:* The area of a triangle given by two complex numbers.
  - We know that each complex number $z = x + iy$ has also a representation with a radius and an angle: radius is $\sqrt{x^2+y^2}$ and angle is $\arctan\left(\frac{y}{x}\right)$ defined up to 2π. Thus we can write $z = (R,\,\varphi)$.
  - We know that multiplying one complex $(R_1,\,\varphi_1)$ number by another $(R_2,\,\varphi_2)$ means rotating z_1 by angle φ_2 counter-clockwise and scaling the radius times R_2.
  - The complex conjugate has the same radius but minus the angle.
  - Thus to find the angle between the two numbers, $\varphi_1 - \varphi_2$, we can just rotate z_1 by $\overline{z}_2$ and measure the angle of that: $\arctan\left(\frac{\Im(z_1\overline{z_2})}{\Re(z_1\overline{z_2})}\right)$.
  - We know that the area of a triangle with two edges a and b and an angle between them α is given by $A = \frac{1}{2}ab\sin(\alpha)$.
  - Recall that $\sin\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{\frac{y}{x}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{y}{\sqrt{x^2+y^2}}$.

1.2 Tips for Specific Questions

1.2.1 Question 3

1. First step: Translate your triangle so that one of the vertices is at 0. A translation of an equilateral triangle is still an equilateral triangle.
2. Secnd step: Scale your triangle by the length of one of the edges, so that that edge will now have length 1, and rotate it so that the edge lies on the x-axis. Such a scaled rotated equilateral triangle will still be an equilateral triangle.
3. What is the condition now for this translated, scaled rotated trinagle to be an equilateral?
4. Convert the condition you obtained back into a condition on z_1, z_2 and z_3.

1.2.2 Question 4

- Separate into cases according to the possible values of a to show that the solutions of the equation necessarily describe straight lines or circles.
- Show that every given straight line or circle can be written using the original equation.

1.2.3 Question 5

- Use trick: $0 = 0 + 0$ and $0 = 1 - 1$.

2 Exercise Sheet Number 2

2.1 Question 1

Let A be a nonempty subset of $\mathbb{R}$. Prove that:

- *Claim:* If A is bounded above, then $-A$ is bounded below and $\inf(-A) = -\sup(A)$.
  Proof:
  - Because:
    - $\mathbb{R}$ has the "supremum property" because we constructed it that way.
    - $A \neq \varnothing$
    - A is bounded above.

    Then we know that $\exists \sup(A) \in \mathbb{R}$. Thus the claim makes sense.
  - Because $\exists \sup(A) \in \mathbb{R}$, we know that
    1. $\forall a \in A : \; a \leq \sup(A)$ and

2. $\nexists x \in \mathbb{R}$ such that
   (a) $x < \sup(A)$.
   (b) $\forall a \in A : a \leq x$.

- This is equivalent (by multiplying all inequalities by -1) to saying that:
  1. $\forall a \in A : -a \geq -\sup(A)$ and
  2. $\nexists x \in \mathbb{R}$ such that:
     (a) $-x > -\sup(A)$.
     (b) $\forall a \in A : -a \geq -x$.
- Define $\alpha := -\sup(A)$.
- This is equivalent (by renaming variables an) to saying that:
  1. $\forall a \in -A : a \geq \alpha$ and
  2. $\nexists x \in \mathbb{R}$such that:
     (a) $x > \alpha$
     (b) $\forall a \in -A : a \geq x$
- This is equivalent to saying that $-A$ is bounded below (by α) and (by the definition of the infimum) to saying that $\exists \inf(-A) \in \mathbb{R}$ and that $\alpha = \inf(-A)$.

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- The second claim is done very similarly.

2.2 Question 2

Let $|M| = |\mathbb{R}|$ and $|A| = |\mathbb{N}|$ such that $A \cap M = \varnothing$.

- *Claim:* $|M \cup A| = |\mathbb{R}|$.
  Proof:
  - $|M| = |\mathbb{R}|$ so that $\exists f : M \to \mathbb{R}$ such that f is bijective.
  - $|A| = |\mathbb{N}|$ so that $\exists g : A \to \mathbb{N}$ such that g is bijective.
  - Define $h : M \cup A \to \mathbb{R}$ by the following rule $h(x) := \begin{cases} f(x) & x \in M \wedge f(x) \notin \mathbb{N} \\ 2f(x) & x \in M \wedge f(x) \in \mathbb{N} \\ 2g(x) + 1 & x \in A \end{cases}$.
  - *Claim:* h is well-defined.
    Proof: For every value in the source set, $M \cup A$, we get only a single value in $\mathbb{R}$.
  - *Claim:* h is surjective.
    Proof:
    * Let $\alpha \in \mathbb{R}$ be given.
    * If $\alpha \notin \mathbb{N}$, then:

- Recall that because f is bijective, $\left|f^{-1}\left(\{\alpha\}\right)\right| = 1$.
- So call this value $\beta \in M$: $h(\beta) = \alpha$ and there must be only *one* such β.

* If $\alpha \in 2\mathbb{N}$, then:
  - Take $f^{-1}\left(\left\{\frac{1}{2}\alpha\right\}\right)$, which is also a set of a single element, call this element $\beta \in M$.
  - So we will have $f(\beta) = \frac{1}{2}\alpha$ and so $h(\beta) = \alpha$.
* If $\alpha \in 2\mathbb{N} + 1$, then:
  - Take $g^{-1}\left(\left\{\frac{\alpha-1}{2}\right\}\right)$. This will be a set of a single element because g is injective.
  - Call this element $\beta \in A$. Then we will have $g(\beta) = \frac{\alpha-1}{2}$ and so $h(\beta) = \alpha$.

– *Claim:* h is injective.
Proof:
  * Let $(x, y) \in (M \cup A)^2$ such that $x \neq y$. We need to show that $h(x) \neq h(y)$.
  * *Case 1*: $x \in M$ and $y \in A$.
    - Then $h(x)$ is either a non-integer or an even number, whereas $h(y)$ is an odd number. In either case, we must have $h(x) \neq h(y)$.
  * *Case 2*: $x \in A$ and $y \in M$ is the same as above by replacing x and y in the statement.
  * *Case 3: $x \in A$ and $y \in A$.*
    - Assume otherwise that $h(x) = h(y)$.
    - Then $2g(x) + 1 = 2g(y) + 1$.
    - Then $g(x) = g(y)$.
    - But then we have $x \neq y$ and $g(x) = g(y)$, contradicting the fact that g is injective.
    - Thus our assumption is false.
  * *Case 4*: $x \in M$ and $y \in M$.
    - *Case 4.1*: $f(x) \notin \mathbb{N}$ and $f(y) \notin \mathbb{N}$. Then because f is injective, $f(x) \neq f(y)$ and so we cannot have that $h(x) = h(y)$.
    - *Case 4.2*: $f(x) \in \mathbb{N}$ and $f(y) \notin \mathbb{N}$. Then $h(x)$ is an even integer and $h(y)$ is not an integer, which means that they cannot be equal.
    - *Case 4.3*: $f(x) \notin \mathbb{N}$ and $f(y) \in \mathbb{N}$ is the same argument as *Case 4.2* but with x and y interchanged.

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2.3 Question 4

- Recall that we are doing two-step induction! So you need to prove for the first induction step both a_0 *and* a_1!
- *Question:* For which values of n is it true that $\left|\frac{a_{n+1}}{a_n} - \phi\right| \leq \varepsilon$ where $\varepsilon > 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$ and $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$ for all $n \in \mathbb{N}$.

  Answer:

  - We start by simplifying the absolute value first:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n} - \phi\right| &= \left|\frac{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]}{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]} - \frac{1}{2}\left(1+\sqrt{5}\right)\right| \\
&= \left|\frac{\left[\frac{1}{2}^{n+2}\left(1+\sqrt{5}\right)^{n+2} - \frac{1}{2}^{n+2}\left(1-\sqrt{5}\right)^{n+2}\right]}{\left[\frac{1}{2}^{n+1}\left(1+\sqrt{5}\right)^{n+1} - \frac{1}{2}^{n+1}\left(1-\sqrt{5}\right)^{n+1}\right]} - \frac{1}{2}\left(1+\sqrt{5}\right)\right| \\
&= \frac{1}{2}\left|\frac{\left[\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}\right]}{\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right]} - \left(1+\sqrt{5}\right)\right| \\
&= \frac{1}{2}\left|\frac{\left[\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}\right]}{\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right]} - \left(1+\sqrt{5}\right)\right| \\
&= \frac{1}{2}\left|1+\sqrt{5}\right|\left|\frac{\left[\left(1+\sqrt{5}\right)^{n+1} - \frac{\left(1-\sqrt{5}\right)^{n+2}}{1+\sqrt{5}}\right]}{\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right]} - 1\right| \\
&= \frac{1}{2}\left|1+\sqrt{5}\right|\left|\frac{\left[\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^{n+1} - \frac{1-\sqrt{5}}{1+\sqrt{5}}\right]}{\left[\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^{n+1} - 1\right]} - 1\right| \\
&= \frac{1}{2}\left|1+\sqrt{5}\right|\left|\frac{-\frac{1-\sqrt{5}}{1+\sqrt{5}} + 1}{\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^{n+1} - 1}\right| \\
&= \frac{\sqrt{5}}{\left|\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^{n+1} - 1\right|}
\end{aligned}$$

  - Next use the fact that $\frac{1+\sqrt{5}}{1-\sqrt{5}} \approx -2.618$.

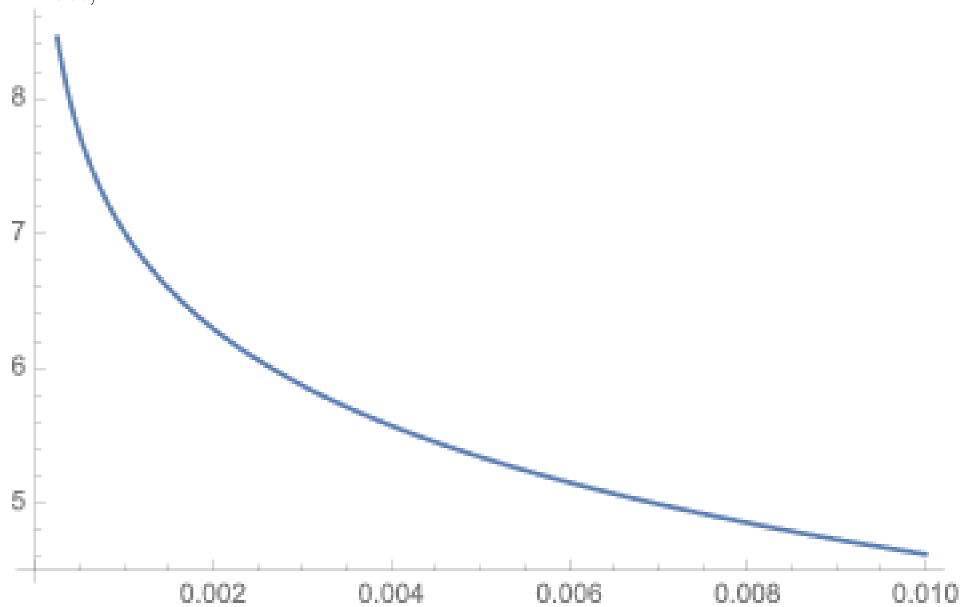
- *Claim:* Reverse triangle inequality: $|x-y| \leq |x|-|y|$.
  Proof:
  * $|x| = |(x-y)+y| \leq |x-y|+|y|$
  * $|x|-|y| \leq |x-y|$
- *Claim:* $\frac{1}{|\alpha^n-1|} \leq \frac{1}{|\alpha|^n-1}$ for all $n \in \mathbb{N}$ and $\alpha < -1$.
  Proof:
  * By the reverse triangle inequality we have $|\alpha^n - 1| \geq |\alpha^n| - |1| = |\alpha|^n - 1$.
  * Because $|\alpha| > 1$, $|\alpha|^n > 1 \forall n \in \mathbb{N}$ and so $|\alpha|^n - 1 > 0$.
  * Thus we may conclude that $\frac{1}{|\alpha^n-1|} \leq \frac{1}{|\alpha|^n-1}$.
- As a result we have that $\frac{\sqrt{5}}{\left|\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^{n+1}-1\right|} \leq \frac{\sqrt{5}}{\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|^{n+1}-1}$ and so $\left|\frac{a_{n+1}}{a_n} - \phi\right| \leq \frac{\sqrt{5}}{\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|^{n+1}-1}$.
- So if we could make sure that $\frac{\sqrt{5}}{\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|^{n+1}-1} \leq \varepsilon$ for some $n \in \mathbb{N}$ then of course $\left|\frac{a_{n+1}}{a_n} - \phi\right| \leq \varepsilon$ as well for that same $n \in \mathbb{N}$.
- This may not be the *smallest* such n because we have made an estimate.
- So we are looking for

$$\begin{aligned} \frac{\sqrt{5}}{\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|^{n+1}-1} &\leq \varepsilon \\ \frac{\sqrt{5}}{\varepsilon} + 1 &\leq \left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|^{n+1} \end{aligned}$$

- Because $\log(x)$ is a monotonically increasing function (as you will prove at some point in the future), $x \leq y$ means $\log(x) \leq \log(y)$.
- Thus we have

$$\begin{aligned} \log\left(\frac{\sqrt{5}}{\varepsilon}\right) &\leq \log\left(\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|^{n+1}\right) \\ \log\left(\frac{\sqrt{5}}{\varepsilon}\right) &\leq (n+1)\log\left(\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|\right) \\ \frac{\log\left(\frac{\sqrt{5}}{\varepsilon}\right)}{\log\left(\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|\right)} - 1 &\leq n \end{aligned}$$

- Try out $\varepsilon = 0.01$: $n \geq \frac{\log\left(\frac{\sqrt{5}}{0.01}\right)}{\log\left(\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|\right)} - 1 = 4.62$.
- In fact,



2.4 Question 6

- *Claim:* $\binom{2n}{n} = \sum_{k=1}^{n} \binom{n}{k} \binom{n}{n-k}$ $\forall n \in \mathbb{N}$.

  Proof:

  - $\binom{n}{k}$ means the number of possible ways to pick k balls out of a row of n balls.
  - To pick n balls out of a row of $2n$ balls, divide your selection process into two stages:
    1. Split the row in two: the first n balls from 1 to n and the second half from $n+1$ to $2n$.
    2. Decide that you will pick k balls from the first half and $n-k$ balls from the second half, so that all together you still have n balls from the entire row.
    3. To pick k balls from the first half there are $\binom{n}{k}$ possibilities.
    4. To pick $n-k$ balls from the second half there are $\binom{n}{n-k}$ possibilities.
    5. Thus all together for this procedure there are $\binom{n}{k}\binom{n}{n-k}$.

  - But there are in general, between $k = 0$ and $k = n$ different ways to do this split-selection process.
  - We must sum up on all the different k possibilities, because they all exist in parallel.

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