Analysis 1
Recitation Session of Week 5

Jacob Shapiro

October 17, 2014

1 General Remarks

- Notation:
  - $A \equiv B$ means A is always defined as B (the writer is trying to say that this definition is standard and she is merely following conventions).
    - Example: $|x+iy| \equiv \sqrt{x^2+y^2}$
    - Use when you want to *recall* a definition that the book / professor / someone else made or makes.
  - $A := B$ means A is right now defiend as B (the writer is making an ad-hoc definition which will make her analysis shorter or more convenient, but this definition's scope is limited to the document in which its written).
    - Example: $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$ ($\mathcal{M}$ is defined an auxilliary symbol instead of having to write $\mathcal{M}_1 \cup \mathcal{M}_2$ all the time).
    - Use when you want to abbreviate operations by a one-letter symbol instead of carrying around repetetively composite objects or definitions.
- For the 1,000,000th time: be careful about the difference between sets and logical statements:
  - A statement is a determination about whether a certain fact is true or false. $x \in A$ is a statement, which can be true or false. When written just like that, as $x \in A$, with no other accompanying text, interpret this as the writer meaning to express: "The fact $x \in A$ is true."
    - To indicate that two statements are equivalent, use the symbol $\boxed{\Longleftrightarrow}$. For example: $[x \in (A \cup B)] \Longleftrightarrow [(x \in A) \vee (x \in B)]$. See how a set-theoretic operation (union) turned into a logical operation (or). There *is* a close connection between the two, but you must be careful with how you combine these operations. Never write, for example, $\left\{\, x \in \mathbb{R} \;\middle|\; x^2 = 4 \,\right\} \wedge \{\, x \in \mathbb{R} \mid x > 0 \,\}$ because that doesn't make any sense!
    - To indicate that one statement follows from the one before, use the symbol $\boxed{\Longrightarrow}$. For example, $x \in A \cap B \Longrightarrow x \in A$.
  - A set is anything that goes inside curly-brakets $\{\ \}$ or otherwise abbreviated by a letter (or letters).
    - Do *not* use logical operations on sets (as in the example above).

2 Exercise Sheet Number 5

2.1 Metric Spaces

Let X be a set. A metric d on X is a map $d : X \times X \to \mathbb{R}$ such that $\forall\, (p,\, q) \in X^2$:

1. $p \neq q \Longrightarrow d(p,\, q) > 0$
2. $d(p,\, p) = 0$
3. $d(p,\, q) = d(q,\, p)$
4. $d(p,\, q) \leq d(p,\, r) + d(r,\, q)$ for any $r \in X$.

- Example:
  Claim: If $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ (that is, $\|v\| \equiv \sqrt{\sum_{j=1}^{n} |v_j|^2}$) then $d(u,\, v) := \|u - v\|$ is a metric on $\mathbb{R}^n$.
  Proof: homework.

2.1.1 Open Balls

Let X be a set with a metric d defined on it, let $x \in X$ and $\varepsilon > 0$. An open ball at x with radius ε, denoted by $B_\varepsilon(x)$, is defined as

$$\boxed{B_\varepsilon(x) := \{\, y \in X \mid d(x,\, y) < \varepsilon \,\}}$$

for Euclidean spaces you can really think of this as the geometric ball, and drawing this for many situations will help your intutition.

2.1.2 Open Sets in a Metric Space

Let X be a set with a metric d defined on it. We define a new property on subsets of metric spaces called being open. The set $U \subseteq X$ is open iff $\forall x \in U \exists \varepsilon > 0 B_\varepsilon (x) \subseteq U$. Denote the set of all open sets on X as $Open(X)$. Thus we have

$$\boxed{Open (X) \equiv \{ U \subseteq X \mid \forall x \in U (\exists \varepsilon > 0 : B_\varepsilon (x) \subseteq U) \}}$$

- Clearly, you can see that X itself is always open and $\varnothing$ is also always open, no matter what X is.
- Homework: prove that arbitrary unions of open sets are open.
- Homework: prove that finite intersections of open sets are open.
- Example:
  Claim: The set (a, b) is open in $\mathbb{R}$ with its usual metric.
  Proof:
  - Let $x \in (a, b)$ be given.
  - Then $a < x < b$.
  - So take the ball around x with $\varepsilon := \frac{1}{2} \min (\{x - a, b - x\})$.
  - This ball is clearly wholy (a, b).

  ■
- Example:
  Claim: The set $[a, b]$ is not open in $\mathbb{R}$ with its usual metric.
  Proof:
  - The condition for openness actually only fails at the end points. So take $a \in [a, b]$.
  - It *is* the case that no matter what ball we take around a, it will always "spill" out of $[a, b]$.
  - Let *any* $\varepsilon > 0$ be given. Then

$$\begin{aligned} B_\varepsilon (a) &\equiv \{ y \in \mathbb{R} \mid |y - a| < \varepsilon \} \\ &= \{ y \in \mathbb{R} \mid -\varepsilon < y - a < \varepsilon \} \\ &= \{ y \in \mathbb{R} \mid a - \varepsilon < y < a + \varepsilon \} \end{aligned}$$

  - So we see that $\left(a - \frac{1}{2}\varepsilon\right) \in B_\varepsilon (a)$ but $\left(a - \frac{1}{2}\varepsilon\right) \notin [a, b]$ obviously for any $\varepsilon > 0$. So we cannot find *any* ball around a which is wholly contained in $[a, b]$.

2.2 Sequences

Let X be a metric space. A sequence, denoted by $(a_n)_{n\in\mathbb{N}}$, is a map from $\mathbb{N} \to X$ such that $n \mapsto a_n$[1].

- Repetition and comparison with sets: Note that it's perfectly possible that there would be a repetition of the value of this sequence for different values of n. In this sense, the order *does* matter, as opposed to sets. For sets, $\{ 1, 1, 1, \dots \} = \{ 1 \}$, that is a set with merely one element, and so we need to retain information about only one 1 and forget that someone crazy once wrote the set with infinite number of 1's. On the other hand, for sequences, we really need the information that 1 appears infinitely many times: this is the information encoded in the sequences.

2.2.1 Convergence of Sequences

> *Definition*: Let a sequence $(a_n)_{n\in\mathbb{N}}$ be given in a metric space X. $(a_n)_{n\in\mathbb{N}}$ is said to *converge* iff $\exists a \in X$ such that $\forall U \in Open(X)$ such that $a \in U$, $\exists m_U \in \mathbb{N}$ such that $[(n \geq m_U) \Longrightarrow a_n \in U]$.

In this case we say that that $(a_n)_{n\in\mathbb{N}}$ converges to a, or that a is the limit of $(a_n)_{n\in\mathbb{N}}$, and sometimes write $\exists \lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} a_n = a$.

- *Claim:* This definition is equivalent to the older definition epsilon-defintion.
  Proof: homework.
  - The older definition style is: $\forall \varepsilon > 0\ \exists m_\varepsilon \in \mathbb{N}$ such that $[(n \geq m_\varepsilon) \Longrightarrow |a - a_n| < \varepsilon]$.

  But it is better to already get used to this definition because it is how we usually do mathematics.

[1]You should observe that $(a_n)_{n\in\mathbb{N}}$ is special notation and if we were to follow our usual notation for maps then we would simply write $a \in X^\mathbb{N}$ for the sequence and $a(n)$ the value of the sequence at n. However, as is usual in math (and physics) we sometimes use some notation that is not strictly necessary just in order to help our memory. If we see the letter a in the middle of a text somewhere, how should we know if it's a set, a map, a group, a vector space, or whatever? However if we see $(a_n)_{n\in\mathbb{N}}$ in the middle of the text we immediately know it is indeed a map, and not just any map, but a special map from $\mathbb{N}$.

- Examples:

1. *Claim:* $\lim_{n\to\infty} n^{-r} = 0$ for all $r \in \mathbb{Q}$ such that $r > 0$.
Proof:
   - Let $U \in Open(\mathbb{R})$ be given. Our goal is to find this special number, $m_U \in \mathbb{N}$ which would give us the special condition, $[(n \geq m_U) \implies n^{-r} \in U]$.
   - The first thing to do is work with open balls, because they are much handier than open sets: because $U \in Open(\mathbb{R})$, we know that $\exists \varepsilon_U > 0$ such that $B_{\varepsilon_U}(0) \subseteq U$. Thus, if we were to show that $n^{-r} \in B_{\varepsilon_U}(0)$, then we will have shown that $n^{-r} \in U$.
   - $n^{-r} \in B_{\varepsilon_U}(0)$ means, by definition, that $|n^{-r} - 0| < \varepsilon_U$ or that $n^{-r} < \varepsilon_U$ (because n^{-r} will always be positive).
   - We can work our way backwards: ${m_U}^{-r} < \varepsilon_U$ means that $m_U > {\varepsilon_U}^{-\frac{1}{r}}$. Thus pick $m_U := ceil\left({\varepsilon_U}^{-\frac{1}{r}}\right) + 1$.
   - As a result we get that really, if $n \geq ceil\left({\varepsilon_U}^{-\frac{1}{r}}\right) + 1$ then $n > {\varepsilon_U}^{-\frac{1}{r}}$ and so $n^{-r} < \varepsilon_U$ and so $n^{-r} \in U$.

   ■

2. More examples in Koenigsberger page 42, along with the actual proofs.
3. *Claim:* $a_n = i^n$ does not converge.
Proof:
   - To prove that a sequence does *not* converge we usually use proof by contradiction.
   - First of all observe that $a_n = i,\, -1,\, -i,\, 1,\, i,\, -1,\, -i,\, 1,\, \dots$ and so on.
   - As a result we see that $|a_l - a_{l+1}| = \sqrt{2}$ for all $l \in \mathbb{N}$.
   - Assume that a_n did converged to some $a \in \mathbb{C}$.
   - Then surely $B_{\sqrt{2}}(a) \in Open(\mathbb{C})$ and $a \in B_{\frac{\sqrt{2}}{2}}(a)$.
   - In that case, we should have that $\exists m_{\frac{\sqrt{2}}{2}} \in \mathbb{N}$ such that $a_n \in B_{\frac{\sqrt{2}}{2}}(a)$ for all $n \geq m_{\frac{\sqrt{2}}{2}}$.
   - This is clearly impossible, because if $a_{m_{\frac{\sqrt{2}}{2}}} \in B_{\frac{\sqrt{2}}{2}}(a)$, then $\left|a - a_{m_{\frac{\sqrt{2}}{2}}}\right| < \frac{\sqrt{2}}{2}$. But then, we have

$$\begin{aligned}
\left|a - a_{m_{\frac{\sqrt{2}}{2}+1}}\right| &= \left|a - a_{m_{\frac{\sqrt{2}}{2}}} + a_{m_{\frac{\sqrt{2}}{2}}} - a_{m_{\frac{\sqrt{2}}{2}+1}}\right| \\
&= \left|\left(a - a_{m_{\frac{\sqrt{2}}{2}}}\right) - \left(a_{m_{\frac{\sqrt{2}}{2}+1}} - a_{m_{\frac{\sqrt{2}}{2}}}\right)\right| \\
&\geq \left|\left|a - a_{m_{\frac{\sqrt{2}}{2}}}\right| - \left|a_{m_{\frac{\sqrt{2}}{2}+1}} - a_{m_{\frac{\sqrt{2}}{2}}}\right|\right| \\
&\geq \underbrace{\left|a_{m_{\frac{\sqrt{2}}{2}+1}} - a_{m_{\frac{\sqrt{2}}{2}}}\right|}_{\sqrt{2}} - \underbrace{\left|a - a_{m_{\frac{\sqrt{2}}{2}}}\right|}_{\leq\frac{\sqrt{2}}{2}} \\
&\geq \sqrt{2} - \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

   that is, $\left|a - a_{m_{\frac{\sqrt{2}}{2}+1}}\right| \geq \frac{\sqrt{2}}{2}$ which means that $a_{m_{\frac{\sqrt{2}}{2}+1}} \notin B_{\frac{\sqrt{2}}{2}}(a)$ and so we have a contradiction.

   ■

2.3 Vector Spaces

2.3.1 Normed Vector Spaces

A normed vector space is tuple $(V,\, \|\cdot\|)$ such that V is a vector space (over a field $\mathbb{F}$[2]) and $\|\cdot\|$ is a norm on V.

- We assume the reader is familiar with "groups" and "fields". In case this is false, please follow up on the definitions in Herstein's "*Topics in Algebra*" 2nd edition chapters two and five.
- A vector space over a field $\mathbb{F}$ is a tuple[3] $V = (V_{set},\, c,\, s)$ such that $(V_{set},\, c)$ is a commutative group (a group where $c((v_1,\, v_2)) = c((v_2,\, v_1))$) and $s \in {V_{set}}^{\mathbb{F}\times V_{set}}$ is a map which obeys the following conditions:

[2] For now think of $\mathbb{F}$ as being $\mathbb{C}$ in the most general case. We will not care about "exotic" fields until a year from now.
[3] V is the object "vector space", V_{set} is its underlying set; we two different symbols to avoid confusion

1. $s \circ c = c \circ (s \times s)$ as an equality of maps[4]. Slightly more explicitly this means $s((\alpha, c((v_1, v_2)))) = c((s((\alpha, v_1)), s((\alpha, v_2))))$ for all $\alpha \in \mathbb{F}$ and for all $(v_1, v_2) \in {V_{set}}^2$.
2. $s \circ (a_\mathbb{F} \times \mathbb{1}_{V_{set}}) = c \circ ((s \circ (\pi_1 \times \pi_3)) \times (s \circ ((\pi_2 \times \pi_3))))$ where π_i is the projection map[5] and $a_\mathbb{F} \in \mathbb{F}^{\mathbb{F}^2}$ is the addition map on $\mathbb{F}$. More explicitly, we require that $s((a_\mathbb{F}(\alpha, \beta), v)) \stackrel{!}{=} c((s((\alpha, v))), s((\beta, v)))$ for all $(\alpha, \beta, v) \in \mathbb{F}^2 \times V_{set}$.
3. $s \circ (\pi_2 \times s \circ (\pi_1 \times \pi_3)) = s \circ (m_\mathbb{F} \times \mathbb{1}_{V_{set}})$ where $m_\mathbb{F}$ is the multiplication map in $\mathbb{F}$ or more explicitly $s((\alpha, s((\beta, v)))) = s((m_\mathbb{F}((\alpha, \beta)), v))$ for all $(\alpha, \beta, v) \in \mathbb{F}^2 \times V_{set}$.
4. $s(1_\mathbb{F}, v) = v$ for all $v \in V_{set}$ where $1_\mathbb{F}$ is the unit element in $\mathbb{F}$.

- A norm $\|\cdot\|$ on a vector space V is a map $V_{set} \to \mathbb{R}$ such that :
  1. $\|s((\alpha, v))\| = |\alpha| \|v\|$ for all $(\alpha, v) \in \mathbb{F} \times V_{set}$.
  2. $\|c((v_1, v_2))\| \leq \|v_1\| + \|v_2\|$ for all $(v_1, v_2) \in {V_{set}}^2$.
  3. $\|v\| = 0 \Longrightarrow v = 0_V$.
- For the sake of brevity from now on we will write αv instead of $s((\alpha, v))$ and $v + v'$ instead of $c((v, v'))$, as well as $\alpha + \beta$ instead of $a_\mathbb{F}((\alpha, \beta))$ and $\alpha\beta$ instead of $m_\mathbb{F}((\alpha, \beta))$, hoping that no confusion will arise.

2.3.2 Inner Product Space

An inner product space is a tuple $(V, \langle\cdot, \cdot\rangle)$ where V is a vector space over a field $\mathbb{F}$ and $\langle\cdot, \cdot\rangle$ is a map from ${V_{set}}^2 \to \mathbb{F}$ such that:

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ where $\overline{\alpha}$ is conjugation in $\mathbb{F}$ (if it exists, otherwise it is the identity) for all $(u, v) \in {V_{set}}^2$.
2. $\langle u, u\rangle \geq 0$ for all $u \in V_{set}$.
3. $\langle u, u\rangle = 0 \Longleftrightarrow u = 0_V$.
4. $\langle \alpha u + \beta v, \omega\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ for all $(\alpha, \beta, u, v, w) \in \mathbb{F}^2 \times {V_{set}}^3$.

Cauchy–Schwarz inequality

- *Claim:* $|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle \langle y, y\rangle}$ for all $(x, y) \in {V_{set}}^2$.
  Proof:
  - Case 1: $x = 0$
    - $\langle x, y\rangle = \langle 0, y\rangle = \langle y - y, y\rangle = \langle y, y\rangle - \langle y, y\rangle = 0$ and clearly $\langle 0, 0\rangle = 0$.
  - Case 2: $x \neq 0$ and $y \neq 0$
    - Define $z := y - \frac{\langle x, y\rangle}{\langle x, x\rangle} x$.
    - Calculate

$$\begin{aligned} \langle x, z\rangle &= \left\langle x, y - \frac{\langle x, y\rangle}{\langle x, x\rangle} x \right\rangle \\ &= \langle x, y\rangle - \frac{\langle x, y\rangle}{\langle x, x\rangle} \langle x, x\rangle \\ &= 0 \end{aligned}$$

[4] If s is a map $s \in B^A$ and $s' \in B'^{A'}$ then the map $s \times s'$ is naturally defined as an element in $B \times B'^{A \times A'}$ via $(s \times s')((a, a')) \equiv (s(a), s'(a'))$ for all $(a, a') \in A \times A'$.

[5] The projection map $\pi_i \in {A_i}^{A_1 \times A_2 \times \cdots \times A_n}$ is defined as $\pi_i((a_1, a_2, \ldots, a_n)) \equiv a_i$.

* Thus we can calculate

$$
\begin{aligned}
\langle y,\, y\rangle &= \left\langle z+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x,\, z+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x\right\rangle \\
&= \langle z,\, z\rangle+\left\langle z,\, \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x\right\rangle+\left\langle \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x,\, z\right\rangle+\left\langle \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x,\, \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x\right\rangle \\
&= \langle z,\, z\rangle+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\underbrace{\langle z,\, x\rangle}_{0}+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\underbrace{\langle x,\, z\rangle}_{\overline{0}}+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\left\langle x,\, \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x\right\rangle \\
&= \langle z,\, z\rangle+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\overline{\left\langle \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}x,\, x\right\rangle} \\
&= \langle z,\, z\rangle+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\overline{\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\langle x,\, x\rangle} \\
&= \langle z,\, z\rangle+\frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\overline{\langle x,\, y\rangle} \\
&\geq \frac{\langle x,\, y\rangle}{\langle x,\, x\rangle}\overline{\langle x,\, y\rangle} \\
&= \frac{|\langle x,\, y\rangle|^2}{\langle x,\, x\rangle}
\end{aligned}
$$

* As a result we find that $\langle y,\, y\rangle\langle x,\, x\rangle \geq |\langle x,\, y\rangle|^2$ and so the result follows by taking the square root of both sides.

2.3.3 Norm out of Inner Product

Given an inner product space, we can easily define a norm out of it in a natural (that is, minimum choices) way:

- Define $\|v\| := \sqrt{\langle v,\, v\rangle}$.
- This makes sense because $\langle v,\, v\rangle \geq 0$ and so $\sqrt{\langle v,\, v\rangle} \in \mathbb{R}$ indeed.
- We can verify that all the conditions on this new induced norm are indeed satisfied (given the axioms of the inner product):
  1. For the homegeneity condition, let $(\alpha,\, v) \in \mathbb{F}\times V_{set}$ be given. Then

$$
\begin{aligned}
\|\alpha v\| &\equiv \sqrt{\langle \alpha v,\, \alpha v\rangle} \\
&= \sqrt{\alpha^2\langle v,\, v\rangle} \\
&= |\alpha|\sqrt{\langle v,\, v\rangle} \\
&\equiv |\alpha|\,\|v\|
\end{aligned}
$$

     as desired.
  2. For the triangle inequality condition, let $(v_1,\, v_2) \in {V_{set}}^2$ be given. Then

$$
\begin{aligned}
\|v_1+v_2\| &\equiv \sqrt{\langle v_1+v_2,\, v_1+v_2\rangle} \\
&= \sqrt{\langle v_1,\, v_1\rangle+\langle v_1,\, v_2\rangle+\langle v_2,\, v_1\rangle+\langle v_2,\, v_2\rangle} \\
&= \sqrt{\langle v_1,\, v_1\rangle+\underbrace{\langle v_1,\, v_2\rangle+\overline{\langle v_1,\, v_2\rangle}}_{\in\mathbb{R}}+\langle v_2,\, v_2\rangle} \\
&= \sqrt{\|v_1\|^2+2\Re\left(\langle v_1,\, v_2\rangle\right)+\|v_2\|^2} \\
&\overset{\Re(z)\leq|z|}{\leq} \sqrt{\|v_1\|^2+2\left|\langle v_1,\, v_2\rangle\right|+\|v_2\|^2} \\
&\overset{C.S.}{\leq} \sqrt{\|v_1\|^2+2\|v_1\|\|v_2\|+\|v_2\|^2} \\
&= \|v_1\|+\|v_2\|
\end{aligned}
$$

  3. For the definitiveness condition, assume for the moment that $\|v\| = 0$ for some $v \in V_{set}$. Our goal is to show that v *must* be equal to 0_V. But $\|v\| \equiv \sqrt{\langle v,\, v\rangle}$, and so we have $\sqrt{\langle v,\, v\rangle} = 0$ or $\langle v,\, v\rangle = 0$. Thus, using the axioms of the inner product we necessarily conclude that $v = 0$.
- As a result we see that every time we encounter an inner product vector space, we can define on top of it also a norm and make it into a normed vector space for free.
- Is the converse also true? No, as we will see soon.

2.3.4 Parallelogram Law

- A normed vector space is said to obey the parallelogram condition iff

$$\boxed{\|u\|^2 + \|v\|^2 = \frac{1}{2}\left(\|u+v\|^2 + \|u-v\|^2\right)\ \forall (u,\, v) \in V_{set}}$$

- An example for a normed vector space that does *not* obey the parallelogram condition:
  - Let $n \in \mathbb{N} \setminus \{0,\, 1\}$.
  - Define a norm on $\mathbb{R}^n$ (which is a bonafide vector space!) by $\|v\|_1 := \sum_{j=1}^{n} |v_j|$.
  - *Claim*: $\|\cdot\|_1$ is indeed a norm.
    Proof:
    1. For the homegeneity condition, let $(\alpha,\, v) \in \mathbb{R} \times \mathbb{R}^n$ be given. Then

    $$\begin{aligned} \|\alpha v\|_1 &\equiv \sum_{j=1}^{n} |\alpha v_j| \\ &= \sum_{j=1}^{n} |\alpha|\,|v_j| \\ &= |\alpha| \sum_{j=1}^{n} |v_j| \\ &= |\alpha|\,\|v\|_1 \end{aligned}$$

       as desired.
    2. For the triangle inequality condition, let $(u,\, v) \in (\mathbb{R}^n)^2$ be given. Then

    $$\begin{aligned} \|u+v\|_1 &\equiv \sum_{j=1}^{n} |u_j + v_j| \\ &\leq \sum_{j=1}^{n} (|u_j| + |v_j|) \\ &= \sum_{j=1}^{n} |u_j| + \sum_{j=1}^{n} |v_j| \\ &= \|u\|_1 + \|v\|_1 \end{aligned}$$

    3. For the definitiveness condition, assume for the moment that $\|v\|_1 = 0$ for some $v \in \mathbb{R}^n$. Our goal is to show that v *must* be equal to 0_V. But $\|v\|_1 \equiv \sum_{j=1}^{n} |v_j|$, and so we have $\sum_{j=1}^{n} |v_j| = 0$. Because every term in the sum is never negative, all the terms must be equal to zero separately: $|v_j| = 0 \forall j \in \{1,\, \dots,\, n\}$. This is exactly the zero vector in $\mathbb{R}^n$ then.

    ■
  - *Claim:* $(\mathbb{R}^n,\, \|\cdot\|_1)$ does not obey the parallelogram condition.
    Proof:
    * Take $u = (3,\, 0,\, \dots,\, 0)$ and $v = (0,\, 4,\, 0,\, \dots,\, 0)$.
    * Then

    $$\begin{aligned} \|u\|_1{}^2 + \|v\|_1{}^2 &= (|3| + |0| + \cdots + |0|)^2 + (|0| + |4| + |0| + \cdots + |0|)^2 \\ &= 9 + 16 \\ &= 25 \end{aligned}$$

    * However,

    $$\begin{aligned} \frac{1}{2}\left(\|u+v\|_1{}^2 + \|u-v\|_1{}^2\right) &= \frac{1}{2}\left(\|(3,\, 4,\, 0,\, \dots,\, 0)\|_1{}^2 + \|(3,\, -4,\, 0,\, \dots,\, 0)\|_1{}^2\right) \\ &= \frac{1}{2}\left[(|3| + |4| + |0| + \cdots + |0|)^2 + (|3| + |-4| + |0| + \cdots + |0|)^2\right] \\ &= (|3| + |4| + \cdots + |0|)^2 \\ &= 49 \end{aligned}$$

    * Of course $25 \neq 49$ and so the parallelogram does not hold for any pair of vectors!

- *Claim:* In an inner product space, the induced norm *always* obeys the parallelogram law.
  Proof: homework! (just plug in the norm expressed in terms of the inner product as defined above and use the axioms of the inner product (linearity)).

- *Claim:* In a normed vector space, if the parallelogram law holds, then there exists an inner product $\langle\cdot,\,\cdot\rangle$ on the vector space such that $\|v\|^2 = \langle v,\, v\rangle$ for all $v \in V_{set}$.
  Proof:
  - Define $\langle u,\, v\rangle := \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$.
  - Now show that this is indeed an inner product, and that the norm induced from this inner product is exactly $\|\cdot\|$.
  - To show additivity, you will need a few tricks, for instance, prove that $\|u+v+w\|^2 = \|w+u\|^2 + \|w+v\|^2 + \|u\|^2 + \|v\|^2 - \|u-v\|^2 - \|w\|^2$ using the parallelogram rule which we assume.
  - To show homogeneity you will have to work a bit harder:
    * First show it holds only for $n \in \mathbb{N}$: $\langle nu,\, v\rangle = n\,\langle u,\, v\rangle$ (using induction).
    * Then show it holds for $n \in \mathbb{Z}$ using a trick: $0 = n + (-n)$ for all $n \in \mathbb{N}$.
    * Then show it holds for $r \in \mathbb{Q}$ by writing $r = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}\backslash\{0\}$. Then calculate what's $q\left\langle \frac{p}{q}u,\, v\right\rangle$ (you may use the homogeneity of the previous steps becasue $q \in \mathbb{N}$!
    * To show it holds for any $\alpha \in \mathbb{R}$, take a sequence of rationals $\alpha_n \in \mathbb{Q}$ such that $\alpha_n \to \alpha$.
      · Use the reverse triangle inequality of norms: Reverse triangle inequality: $|\|x\| - \|y\|| \leq \|x-y\|$. *Proof:*
      1. $\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|$
      2. $\|x\| - \|y\| \leq \|x-y\|$
      3. By symmetry we have $\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$, that is $\|x\| - \|y\| \geq -\|x-y\|$
    * Thus

$$\begin{aligned}
|\|\alpha_n v + w\| - \|\alpha v + w\|| &\leq \|\alpha_n v + w - \alpha v - w\| \\
&= \|\alpha_n v - \alpha v\| \\
&= \|(\alpha_n - \alpha)\, v\| \\
&\overset{\text{homogen}}{=} |\alpha_n - \alpha|\, \|v\|
\end{aligned}$$

    * Then try to compute $\langle \alpha u,\, v\rangle = \langle (\lim_{n\to\infty}\alpha_n)\, u,\, v\rangle$ using the definition of $\langle\cdot,\,\cdot\rangle$.

2.4 Concrete Tips for the Questions

2.4.1 Question 2

- May be helpful to use the formula $(a \times b)_i = \sum_{j=1}^{n}\sum_{k=1}^{n}\varepsilon_{ijk}a_j b_k$ where $\varepsilon_{ijk} \equiv \begin{cases} 1 & (i,\,j,\,k) \in A_n \\ -1 & (i,\,j,\,k) \in S_n \backslash A_n \\ 0 & (i,\,j,\,k) \notin S_n \end{cases}$ where S_n is the set of all bijections on the set $\{1,\,\ldots,\,n\}$ ($\left\{ (\sigma_1,\,\ldots,\,\sigma_n) \in \{1,\,\ldots,\,n\}^n \,\middle|\, \sigma_i \neq \sigma_j \forall\,(i,j) \in \{1,\,\ldots,\,n\}^2 \text{ such that } i \neq j \right\}$) and A_n is the set of all bijections on the set $\{1,\,\ldots,\,n\}$ which are even (which take an even number of transpositions to produce the ascending order of $\{1,\,\ldots,\,n\}$).
  - Example: $A_3 = \{\,(1,\,2,\,3)\,,\,(2,\,3,\,1)\,,\,(3,\,1,\,2)\,\}$, and $S_3 = \{\,(1,\,2,\,3)\,,\,(2,\,3,\,1)\,,\,(3,\,1,\,2)\,,\,(3,\,2,\,1)\,,\,(1,\,3,\,2)\,,\,(2,\,1,\,3)\,\}$.
  - Thus $\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$ if $n = 3$.
  - Why is this helpful? Because then:

$$\begin{aligned}
\left[(a + a') \times b\right]_i &= \sum_{j,\,k}\varepsilon_{ijk}\,(a+a')_j\, b_k \\
&= \sum_{j,\,k}\varepsilon_{ijk}a_j b_k + \sum_{j,\,k}\varepsilon_{ijk}a'_j b_k \\
&= [a \times b]_i + [a' \times b]_i
\end{aligned}$$

    due to the linearity of operations in the underlying field.
  - To deduce the other properties you need to think about some facts about ε_{ijk}:
    1. What's $\sum_{i=1}^{3}\varepsilon_{ijk}\varepsilon_{imn}$? This will turn out to be useful.
    2. Is ε_{ijk} symmetric under exchange of any two indices?

2.4.2 Question 3

- Try to make a proof of each of the questions, unless you think the statement is false, in which case give a counter example! This question is really merely about plugging in the definitions and includes no creativity.

2.4.3 Question 4

- Draw a picture of the situation, use your geometric intutition to guess which vector v in E would minimize the set $\{ \|v - a\| \mid v \in E \}$. Prove that this particular vector indeed minimizes the set by picking another arbitrary vector v' in E and showing that necessarily $\|v' - a\| \geq \|v - a\|$.

3 Exercise Sheet Number 3

3.1 Question 1

- *Claim*: If $f \in Y^X$ and $(A, B) \in 2^Y$ then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
  Proof:

3.2 Question 3

- *Claim*: $\left|\left\{ A \in 2^{\mathbb{N}} \mid |A| \in \mathbb{N} \right\}\right| = \mathbb{N}$.
  Proof:
  - Define $\varphi : \left\{ A \in 2^{\mathbb{N}} \mid |A| \in \mathbb{N} \right\} \to \mathbb{N}$ by $\{n_1, \ldots, n_k\} \mapsto \sum_{i=1}^{k} 2^{n_i}$ and $\varnothing \mapsto 0$.
  - This is a surjection, because any integer can be represented as a binary number, which is, in turn, taking taking the digits present in the number and placing them in our set. Alternatively, it is injective because if two sets are equal, they will necessarily define the same binary number ($\sum_{i=1}^{k} 2^{n_i}$ is injective!)

  ■