Analysis 1
Recitation Session of the 6th Week

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Abstract

For the exercises in homework number 6 I especially recommend to review chapter 3 in Rudin's Principles of Mathematical Analysis (PMA). The presentation is extremely consice and useful. The topic of the next colloquium will be limit points of a sequence and discussion of remaining questiosn on the topic of sequences.

1 Exercise Sheet Number 6

1.1 Convergence in Metric Spaces

1.1.1 The Definition of Convergence in Metric Spaces

Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

We say that $(x_n)_{n\in\mathbb{N}}$ converges in X iff:
$\exists\alpha \in X$ such that $\forall\varepsilon > 0$, $\exists m_\varepsilon \in \mathbb{N}$ such that $[n \geq m_\varepsilon \Longrightarrow d(x_n, \alpha) < \varepsilon]\, \forall n \in \mathbb{N}$.
In this case α is called the limit of $(x_n)_{n\in\mathbb{N}}$.

1.1.2 Subsequences

Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. A subsequence of $(x_n)_{n\in\mathbb{N}}$ is specified by a subset $S \subseteq \mathbb{N}$, and written as $(x_n)_{n\in S}$. Then there can be situations where a sequence does *not* converge yet a subsequence of it does. Convergence of a subsequence is defien as:

We say that the subsequence $(x_n)_{n\in S}$ converges in X iff:
$\exists\alpha \in X$ such that $\forall\varepsilon > 0$, $\exists m_\varepsilon \in S$ such that $[n \geq m_\varepsilon \Longrightarrow d(x_n, \alpha) < \varepsilon]\, \forall n \in S$.
In this case α is called the limit of $(x_n)_{n\in S}$.

- Example: $((-1)^n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$ but $((-1)^n)_{n\in 2\mathbb{N}}$ does.

1.1.3 Lim Sup and Lim Inf

Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence[1] in X.

Define a new set $E := \{\, x \in X \mid \exists S \subseteq \mathbb{N} : (x_n)_{n\in S} \to x \,\}$.
Define $\boxed{\limsup\limits_{n\to\infty} x_n := \sup(E)}$ and $\boxed{\liminf\limits_{n\to\infty} x_n := \inf(E)}$.

- So this gives the supremum, or infimum of all possible limits from *sub*sequences.
- The lim inf and lim sup exist even when the sequence doesn't actually converge.
- For a real-valued sequence $(x_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} x_n = s$ iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} = s$.
- Example: If $x_n = \frac{(-1)^n}{1+\frac{1}{n}}$ then $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} = -1$.
- Theorem 3.19 in Rudin's PMA:
  - If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences, and if $\exists N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \in \mathbb{N}$ such that $n \geq N$ then $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$ and $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty}$. This is an important tool that is called "taking the lim inf" or "taking the lim sup" of both sides of an inequality.

[1] There is an added complication about subsequences that diverge to infinity. Those who need the details can follow Rudin's PMA chapter 3.

1.1.4 Series

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Then we can create a new sequence out of it: $\left(\sum_{j=1}^{n} x_j\right)_{n\in\mathbb{N}}$. Clearly, $\left(\sum_{j=1}^{n} x_j\right)_{n\in\mathbb{N}}$ is also a sequence. $\left(\sum_{j=1}^{n} x_j\right)_{n\in\mathbb{N}}$ is called then a series and usually instead of writing $\lim_{n\to\infty}\sum_{j=1}^{n} x_j$ one writes $\sum_{j=1}^{\infty} x_j$. There is really nothing new to say about series: all the theorems and properties of sequences carry over to series, and in order to prove that a series convergence, one has to merely use the theorems and techniques of sequences, but applied to this new constructed sequence (which we call series). Really, the term 'series' exists merely for human convenience.

- Note that the sequence that we started with, $(x_n)_{n\in\mathbb{N}}$ may very well converge whereas its corresponding series $\left(\sum_{j=1}^{n} x_j\right)_{n\in\mathbb{N}}$ will not. Example: $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges yet $\left(\sum_{j=1}^{n}\frac{1}{j}\right)_{n\in\mathbb{N}}$ does not!

1.2 Some Examples

1.2.1 e

- *Definition*: $e := \sum_{j=0}^{\infty}\frac{1}{j!}$, where $0! \equiv 1$.
- *Claim:* $\sum_{j=0}^{\infty}\frac{1}{j!}$ converges in $\mathbb{R}$.
  Proof:
  - We can show convergence using Theorem 3.24 in Rudin's PMA: "A series of nonnegative terms converges iff its partial sums form a bounded sequence."
  - Clearly $\frac{1}{j!}$ are nonnegative and so we may apply the theorem.
  - The partial sums are given by $\sum_{j=0}^{n}\frac{1}{j!}$. Our goal is to show that $\left(\sum_{j=0}^{n}\frac{1}{j!}\right)_{n\in\mathbb{N}}$ is a bounded sequence (that is, that $\exists M\in\mathbb{R}$ such that $\sum_{j=0}^{n}\frac{1}{j!} < M$ for all $n\in\mathbb{N}$).
  - Note that $\sum_{j=0}^{n}\frac{1}{j!} < 1 + \sum_{j=0}^{n}\frac{1}{2^j}$ because

$$\begin{aligned}
\sum_{j=0}^{n}\frac{1}{j!} &= \frac{1}{1}+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{1\cdot 2\cdot\cdots\cdot n} \\
&< \frac{1}{1}+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 2}+\cdots+\frac{1}{1\cdot 2\cdot\cdots\cdot 2} \\
&= 1+1+\frac{1}{2}+\frac{1}{2\cdot 2}+\cdots+\frac{1}{\underbrace{2\cdot\cdots\cdot 2}_{n-1\text{ times}}} \\
&= 1+\sum_{j=0}^{n-1}\frac{1}{2^j}
\end{aligned}$$

  - But now we can use the formula $\sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^j = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2-\left(\frac{1}{2}\right)^{n-1}$, and so $\sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^j < 2$ for all $n\in\mathbb{N}$.
  - But then $\sum_{j=0}^{n}\frac{1}{j!} < 3$ for all $n\in\mathbb{N}$, and so, the sequence is bounded!
  - So using the aforementioned theorem, $\left(\sum_{j=0}^{n}\frac{1}{j!}\right)_{n\in\mathbb{N}}$ converges, and the value it converges to is *defined* as e.

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We can also give another definition of e.

- *Claim*: $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ to e.
  Proof:
  - *Claim:* $\left(1+\frac{1}{n}\right)^n \le \sum_{j=0}^{n}\frac{1}{j!}$ for all $n\in\mathbb{N}$.
    Proof:

* By the binomial theorem,

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{j=0}^{n}\binom{n}{j}\frac{1}{n^j} \\
&= \sum_{j=0}^{n}\frac{n!}{j!\,(n-j)!}\frac{1}{n^j} \\
&\overset{n!\equiv n[(n-1)!]}{=} \sum_{j=0}^{n}\frac{n\,(n-1)\,(n-2)\ldots[n-(j-1)]}{j!}\frac{1}{n^j} \\
&= \sum_{j=0}^{n}\frac{1}{j!}\frac{n\,(n-1)\,(n-2)\ldots[n-(j-1)]}{n^j} \\
&= \sum_{j=0}^{n}\frac{1}{j!}\frac{n\,(n-1)\,(n-2)\ldots[n-(j-1)]}{n\cdot n\cdot n\cdot\cdots\cdot n} \\
&= \sum_{j=0}^{n}\frac{1}{j!}\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-(j-1)}{n} \\
&= \sum_{j=0}^{n}\frac{1}{j!}\underbrace{\left(1-\frac{1}{n}\right)}_{<1}\underbrace{\left(1-\frac{2}{n}\right)}_{<1}\cdots\underbrace{\left(1-\frac{j-1}{n}\right)}_{<1} \\
&< \sum_{j=0}^{n}\frac{1}{j!}
\end{aligned}$$

- Because $\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}$ exists and $\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}=e$ then $\limsup_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}=e$.
- But by theorem 3.19 in Rudin's PMA, and the above inequality, we have that $\limsup_{n\to\infty}\left(1+\frac{1}{n}\right)^n\le e$.
- Let $m\in\mathbb{N}$ be given.
- If $n\ge m$ then

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{j=0}^{n}\frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right) \\
&= \sum_{j=0}^{m}\frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)+\sum_{j=m+1}^{n}\frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right) \\
&\ge \sum_{j=0}^{m}\frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)
\end{aligned}$$

- We may take the $\liminf$ of both sides (this is always allowed) to get:

$$\begin{aligned}
\liminf_{n\to\infty}\left(1+\frac{1}{n}\right)^n &\ge \sum_{j=0}^{m}\frac{1}{j!}\liminf_{n\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)\right] \\
&= \sum_{j=0}^{m}\frac{1}{j!}\left[\liminf_{n\to\infty}\left(1-\frac{1}{n}\right)\right]\left[\liminf_{n\to\infty}\left(1-\frac{2}{n}\right)\right]\cdots\left[\liminf_{n\to\infty}\left(1-\frac{j-1}{n}\right)\right] \\
&= \sum_{j=0}^{m}\frac{1}{j!}1\cdot 1\cdot\cdots\cdot 1 \\
&= \sum_{j=0}^{m}\frac{1}{j!}
\end{aligned}$$

- But $m\in\mathbb{N}$ was arbitrary, so that we may take the $\liminf_{m\to\infty}$ of both sides. The left hand side doesn't *have* any m dependence so that it is actually just a constant. The right hand side is exactly our definition of e, which, because it converges, is equal to its $\liminf$. So that we get:

$$\liminf_{n\to\infty}\left(1+\frac{1}{n}\right)^n \ge e$$

- Recall that $\liminf$ was the infimum of some set E and $\limsup$ was the supremum of E. As such, $\inf(E)\le\sup(E)$ no matter what E is.

- But we have $\liminf_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq e$ and $\limsup_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e$.
- Thus $\sup(E) \leq e$ and $\inf(E) \geq e$. Combining everything together we get that $\sup(E) \leq e$ and $\sup(E) \geq e$ which means that $\sup(E) = e$ and likewise $\inf(E) = e$.
- From this it follows directly that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and $\left(1+\frac{1}{n}\right)^n = e$.

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1.3 Specific Tips for the Exercise Sheet

1.3.1 Question 1

- Use $(a-b)(a+b) = a^2 - b^2$.

1.3.2 Question 2

- Let $\varepsilon > 0$ be given.
- Because $(a_n)_{n\in\mathbb{N}}$ converges (let's say to some a), $\exists m_\varepsilon \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq m(\varepsilon)$.
- Define $M(\varepsilon) := \max(\{\, |a_n - a| \mid n \in \{1, 2, \ldots, m(\varepsilon)\}\,\})$.
- Now try to prove the convergence of the actual thing, by making some estimates. You will have to "take the lim sup" of both sides of the equation at one point.
- For (b), try out some examples with some sequences you know which don't converge.

1.3.3 Question 4

- All but the triangle inequality axiom are trivial.
- Let $p \in \mathbb{N}$ be given. Define $d_p : \mathbb{N}^2 \to \mathbb{R}$ by: $d_p((m, n)) := \begin{cases} 0 & m = n \\ \min\left(\left\{\, p^{-k} \in \mathbb{Q} \;\middle|\; k \in \mathbb{N} \wedge p^k | (m-n) \,\right\}\right) & m \neq n \end{cases}$. The minimum is taken over the set of all p^{-k} where k ranges over all natural numbers which satisfy $p^k | (m-n)$.
- For the triangle inequality, take out the highest power of p out of the two numbers you are comparing $(m-n)$ and $(m-k)$. What is $d(m, n)$ or $d(m, k)$ in terms of these powers? Take the minimum of these powers out of $(n-k) = (n-m) + (m-k)$. What should $d(n, k)$ bet in relation to this power?

1.3.4 Question 5

- For part (a), "follow your nose". This is a question where you just plug in the definitions without employing creaitivyt.
- For part (b), prove the converse of the statement if that's easier for you.

1.3.5 Question 6

- Use the various convergence tests (see Rudin's PMA pages 65-69 "The root and ratio test"). Theorems 3.24 and 3.25 in the same book might also prove useful.

2 Exercise Sheet Number 4

2.1 General Remarks

- It's not *always* necessary to work with real values. That is, if you get an equation of complex numbers, don't *automatically* convert it into an equation with only real values. Sometimes that just generates more work. For example, $|z - m| = r$. You don't need to translate this to $z = x + iy$ to understand what it means geometrically. List of geometric operations directly on complex numbers:
  1. Think of $z \in \mathbb{C}$ like an arrow in $\mathbb{R}^2$: something that has a magnitude and a direction (angle).
  2. $z + z_0$ moves z by the arrow z_0.
  3. $z_0 z$ scales z by $|z_0|$ and rotates z by $\arg(z_0)$.
  4. $|z|$ gives you the magnitude of z.
  5. $\overline{z}$ flips the angle to the negative.

2.2 Question 4

- Part (a):
  - *Claim:* The equation $a|z|^2 + bz + \overline{bz} + c = 0$ describes all the circles and straight lines in the plane, where $(a, c) \in \mathbb{R}^2$ and $b \in \mathbb{C}$ such that $|b|^2 - 4ac > 0$.
    Proof:
    * Case 1: $a = 0$:
      - If $a = 0$, we get $bz + \overline{bz} + c = 0$ with $|b|^2 > 0$ which means $b \neq 0$ and $c \in \mathbb{R}$ is unconstrained.
      - This clearly describes a general line in the plane. To see this,

$$\begin{aligned} bz + \overline{bz} + c &= 0 \\ [\Re(b) + i\Im(b)][\Re(z) + i\Im(z)] + [\Re(b) - i\Im(b)][\Re(z) - i\Im(z)] + c &= 0 \\ 2\Re(b)\Re(z) - 2\Im(b)\Im(z) + c &= 0 \end{aligned}$$

      which is clearly a *general* straight line in $\mathbb{R}^2$ because $b \in \mathbb{C}\setminus\{0\}$ and $c \in \mathbb{R}$ are unconstrained.
    * Case 2: $a \neq 0$:
      - Because $a \neq 0$, we may divide through a to obtain $|z|^2 + \frac{b}{a}z + \frac{\overline{b}}{a}\overline{z} + \frac{c}{a} = 0$.
      - Thus we obtain

$$\begin{aligned} x^2 + y^2 + 2\frac{\Re(b)}{a}x - 2\frac{\Im(b)}{a}y &= -\frac{c}{a} \\ \left(x + \frac{\Re(b)}{a}\right)^2 + \left(y - \frac{\Im(b)}{a}\right)^2 &= -\frac{c}{a} + \left[\frac{\Re(b)}{a}\right]^2 + \left[\frac{\Im(b)}{a}\right]^2 \\ \left(x + \frac{\Re(b)}{a}\right)^2 + \left(y - \frac{\Im(b)}{a}\right)^2 &= -\frac{c}{a} + \frac{|b|^2}{a^2} \\ \left(x + \frac{\Re(b)}{a}\right)^2 + \left(y - \frac{\Im(b)}{a}\right)^2 &= \frac{|b|^2 - ac}{a^2} \end{aligned}$$

      Because we were given that $|b|^2 - ac > 0$ this is indeed a circle.
      - We only have left to show that it is a *generic* circle. By that we mean that *any* circle centered at (x_0, y_0) with radius r such that $r > 0$ can be expressed in this way using the three parameters $(a, c) \in \mathbb{R}^2$ such that $a \neq 0$ and $b \in \mathbb{C}$.
      - So pick $a = 1$, $b = -x_0 + iy_0$ and $c = r^2 - {x_0}^2 - {y_0}^2$ to produce the desired circle.

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- Part (b)
  - Define a map $I \in \mathbb{C} \cup \{\infty\}^{\mathbb{C}\cup\{\infty\}}$ by $z \overset{I}{\mapsto} \begin{cases} \infty & z = 0 \\ 0 & z = \infty \\ \frac{1}{z} & z \in \mathbb{C}\setminus\{0\} \end{cases}$.
  - *Claim:* I sends straight lines and circles into straight lines and circles.
    Proof:
    * Let a straight-line-or-circle be given. By part (a) we know that this corresponds to some choice of $(a, c) \in \mathbb{R}^2$ and $b \in \mathbb{C}$ such that $|b|^2 - 4ac > 0$, and further, that any point z on the straight-line-or-circle will obey the equation $a|z|^2 + bz + \overline{bz} + c = 0$. That is, the straight-line-or-circle will be the set $S_{a,\,b,\,c} := \left\{ z \in \mathbb{C} \cup \{\infty\} \,\middle|\, a|z|^2 + bz + \overline{bz} + c = 0 \right\}$.
    * For I to send a straight-line-or-circle into a straight-line-or-circle, we need the image set $I(S_{a,\,b,\,c})$ to be equal to $I(S_{a,\,b,\,c}) = \left\{ z \in \mathbb{C} \cup \{\infty\} \,\middle|\, \tilde{a}|z|^2 + \tilde{b}z + \overline{\tilde{b}z} + \tilde{c} = 0 \right\}$ for some $(\tilde{a}, \tilde{c}) \in \mathbb{R}^2$ and $\tilde{b} \in \mathbb{C}$ such that $\tilde{b}^2 - \tilde{a}\tilde{c} > 0$ (by part (a), because that equation will describe *any* straight-line-or-circle).
    * Recall that if $f : X \to Y$ and $A \subseteq X$ then $f(A) \equiv \{ f(x) \in Y \mid x \in A \}$.
    * So to show that, let's compute the image set $I(S_{a,\,b,\,c})$:

$$\begin{aligned} I(S_{a,\,b,\,c}) &= \{ I(z) \in \mathbb{C} \cup \{\infty\} \mid z \in S_{a,\,b,\,c} \} \\ &= \left\{ \frac{1}{z} \in \mathbb{C} \cup \{\infty\} \,\middle|\, a|z|^2 + bz + \overline{bz} + c = 0 \right\} \\ &\overset{\frac{1}{z\overline{z}}}{=} \left\{ \frac{1}{z} \in \mathbb{C} \cup \{\infty\} \,\middle|\, a + b\frac{1}{\overline{z}} + \overline{b}\frac{1}{z} + \frac{c}{|z|^2} = 0 \right\} \\ &= \left\{ z \in \mathbb{C} \cup \{\infty\} \,\middle|\, c|z|^2 + \overline{b}z + b\overline{z} + a = 0 \right\} \end{aligned}$$

* $\ast$ So that we indeed get that $I(S_{a,b,c}) = \left\{ z \in \mathbb{C} \cup \{\infty\} \,\middle|\, \tilde{a}|z|^2 + \tilde{b}z + \overline{\tilde{b}z} + \tilde{c} = 0 \right\}$ for some $(\tilde{a}, \tilde{c}) \in \mathbb{R}^2$ and $\tilde{b} \in \mathbb{C}$, where $\tilde{a} = c$, $\tilde{b} = \overline{b}$, $\tilde{c} = a$.
* $\ast$ We only have left to verify that $\tilde{b}^2 - \tilde{a}\tilde{c} > 0$:

$$\begin{aligned} \tilde{b}^2 - \tilde{a}\tilde{c} &= \tilde{b}\overline{\tilde{b}} - ca \\ &= \overline{b}b - ca \\ &= |b|^2 - ac \\ &> 0 \end{aligned}$$

indeed.

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- Part (c)
  - From the above two proofs we see that
    * If $a = 0$ and $c = 0$ (i.e. we started with a line that went through the origin), then $\tilde{a} \equiv c = 0$ and so we will again get a line.
    * If $a = 0$ and $c \neq 0$ (i.e. we started with a line that does not go through the origin), then $\tilde{a} \equiv c \neq 0$ and so we will get a circle!
    * If $a \neq 0$ then we start from a cricle. Then,
      · If $c = 0$ we get a line.
      · If $c \neq 0$ we get a circle again.
  - With these considerations we can solve part (c):
    * (i) and (ii) are two lines that go through the origin and so they will also go be straight lines that go through the origin. To get the actual image we must compute it!
    * (iv) is a line that does not go through the origin, and so we will get a circle.
    * (iii) is a circle, and in particular, it's a circle with $c \neq 0$ (because $b = 0$) and so we will get a circle again.
  - To actually compute it, take a random element parametrizing your set, for instance, for $|z| = r$, parameterize it with re^{it} where $t \in [0, 2\pi)$. Now apply the map on it to get $r^{-1}e^{-it}$. So we get a circle with the radius $\frac{1}{r}$ and going in the opposite direction (but that doesn't matter because sets don't care about order).

2.3 Question 5

- *Claim:* $0 \cdot v = 0_V$ for all $v \in V$ where V is some vector space, and 0_V is its neutral vector under vector addition.
  Note: Some people solve this for the case where $V \simeq \mathbb{C}^n$. But this is not a valid proof because you vector space may very well be infinite dimensional!
  Proof:
  - $0 \cdot v = (0+0) \cdot v \overset{\text{dist. axiom in } V}{=} 0 \cdot v + 0 \cdot v$.
  - But since V is a vector space (and thus a group) there must an additive-inverse to $0 \cdot v$, which is denoted as $-0 \cdot v$.
  - Thus $0_V \equiv 0 \cdot v - 0 \cdot v = 0 \cdot v + 0 \cdot v - 0 \cdot v = 0 \cdot v$.

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- *Claim:* $(-1) \cdot v = -v$, that is, $(-1) \cdot v$ is the additive inverse of v, for any $v \in V$.
  Proof:
  - $(-1) \cdot v + v \overset{\text{dist. axiom in } V}{=} (-1+1) \cdot v = 0 \cdot v \overset{\text{first part}}{=} 0_V$.
  - Thus $(-1) \cdot v$ is the bonda-fide additive-inverse of v, and so we may rightfully write $-v = (-1) \cdot v$.

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