Analysis 1
Recitation Session of Week 7

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Abstract

The topic for next week's colloquium is "space filling curves". The basis for this would be chapter 44 in James Munkres *Topology* (2nd edition) pages 272-275.

1 Exercise Sheet Number 5

1.1 General Remarks

- Many people didn't try the second part of question 1: Just beause I say something is hard doesn't mean you shouldn't try it!
- Define the cross product via levi-civita tensor and thus make life easier: $(x \times y)_i \equiv \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} x_j y_k$ will give you all the properties you want.
- Do *not* need to plug in $z = x + iy$ whenever you see a complex number! (especially for question 3).
- Always think about the "weird" cases, for example, when checking homogeneity, what about $\lambda \in \mathbb{C}$? If you don't care about "weird" cases and you think this is just being fussy, then you're in the wrong field of study! (question 3)
- Do not hand in solutions without proofs.
- Can only use the algebraic laws of limits IF constituent limits exist! Algebraic laws of limits do *not* include roots. For that you need to show that the root function is continuous. Cannot do $\infty - \infty = 0$!
- A very useful too to prove convergence is $a_n \leq b_n$. Use it!

1.2 Question 1 (The Jordan von Neumann Theorem)

- Remember that inner product is not necessarily real! That means in the most general case you can only assume that $\langle u, v\rangle = \overline{\langle v, u\rangle}$ and not $\langle u, v\rangle = \langle v, u\rangle$ (which holds only if the inner product is real).
  - Linearity also follows then *only* in the first argument: $\langle \alpha u + \beta v, \omega\rangle = \alpha \langle u, w\rangle + \beta \langle v, w\rangle$ for all $(\alpha, \beta, u, v, w) \in \mathbb{F}^2 \times {V_{set}}^3$. And for the second argument you have conjugate-linearity: $\langle u, \alpha v + \beta w\rangle = \overline{\alpha} \langle u, v\rangle + \overline{b} \langle u, w\rangle$.
- Need to define the inner product induced by the norm differently then for the complex case!
  - $\langle u, v\rangle := \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$ only gives you a *real* inner product. There is a more general way to get a complex inner product: $\langle u, v\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n \|u + i^n v\|^n$. We will, however, ignore this for the sake of this exercise to simplify things a bit.

1.2.1 How the Heck to Show Additivity??

- Show $\langle n\,u, v\rangle = n\langle u, v\rangle$ for all $n \in \mathbb{N}$ using induction.
- Show $\langle n\,u, v\rangle = n\langle u, v\rangle$ for all $n \in -\mathbb{N}$ using:
  - $0 = \langle 0\,u, v\rangle = \langle (n-n)\,u, v\rangle = \langle nu, v\rangle + \left\langle \underbrace{(-n)}_{\text{positive}}\; u, v\right\rangle = \langle nu, v\rangle + (-n)\langle u, v\rangle$ and so $\langle nu, v\rangle = n\langle u, v\rangle$ for all $n \in -\mathbb{N}$.
- Show $\langle ru, v\rangle = r\langle u, v\rangle$ for all $r \in \mathbb{Q}$ by writing $r = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}\setminus\{0\}$. Then $q\langle ru, v\rangle = q\left\langle \frac{p}{q}u, v\right\rangle = \langle pu, v\rangle = p\langle u, v\rangle$ and so $\langle ru, v\rangle = r\langle u, v\rangle$.
- Show the Cauchy-Schwarz inequality holds still before we proved that $\langle\cdot, \cdot\rangle$ is an inner product (usually the proof hinges on additivity): $|\langle u, v\rangle| \leq \|u\|\|v\|$.
  Proof:

- If $\langle u,\, v\rangle = 0$ the theorem holds trivially and we are done. Otherwise, we may freely assume that $u \neq 0$ and $v \neq 0$.
- Let $r \in \mathbb{Q}$ be given.
- Then

$$\begin{aligned}
0 &\leq \|ru - v\|^2 \\
&= \langle ru - v,\, ru - v\rangle \\
&= \langle ru,\, ru - v\rangle - \langle v,\, ru - v\rangle \\
&= r\,\langle u,\, ru - v\rangle - \langle v,\, ru - v\rangle \\
&= r\overline{\langle ru - v,\, u\rangle} - \overline{\langle ru - v,\, v\rangle} \\
&= r\overline{\langle ru,\, u\rangle - \langle v,\, u\rangle} - \overline{\langle ru,\, v\rangle - \langle v,\, v\rangle} \\
&= rr\overline{\langle u,\, u\rangle} - r\overline{\langle v,\, u\rangle} - \overline{r\,\langle u,\, v\rangle} + \overline{\langle v,\, v\rangle} \\
&= |r|^2\,\|u\|^2 - r\,\langle u,\, v\rangle - \overline{r}\overline{\langle u,\, v\rangle} + \|v\|^2 \\
&= |r|^2\,\|u\|^2 - 2\Re\left(r\,\langle u,\, v\rangle\right) + \|v\|^2
\end{aligned}$$

  for any $r \in \mathbb{Q}$.
- Because for the scope of this discussion we have constructed a real inner product, we can safely write

$$0 \leq r^2\|u\|^2 - 2r\,\langle u,\, v\rangle + \|v\|^2$$

  for any $r \in \mathbb{Q}$.
- So take a sequence of rationals r_n which converges to $\frac{\langle u,\, v\rangle}{\|v\|^2}$.
- Then we have

$$\begin{aligned}
0 &\leq \left(\frac{\langle u,\, v\rangle}{\|u\|^2}\right)^2\|u\|^2 - 2\left(\frac{\langle u,\, v\rangle}{\|u\|^2}\right)\langle u,\, v\rangle + \|v\|^2 \\
0 &\leq |\langle u,\, v\rangle|^2 - 2\,|\langle u,\, v\rangle|^2 + \|v\|^2\|u\|^2
\end{aligned}$$

  from which our result follows.

- Now finally show $\langle \alpha u,\, v\rangle = \alpha\,\langle u,\, v\rangle$ for all $\alpha \in \mathbb{R}$ by:
  - Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of *rational* numbers converging to α (we can always find something like that, for instance, take the decimal expansion of α and always truncate after a finite number of digits).
  - Then what we want to show is $\langle \lim_{n\to\infty} a_n u,\, v\rangle = \lim_{n\to\infty}\langle a_n u,\, v\rangle$ which would give our desired result immediately using all the above steps because $\lim_{n\to\infty}\langle a_n u,\, v\rangle = \lim_{n\to\infty} a_n\,\langle u,\, v\rangle = \alpha\,\langle u,\, v\rangle$.
  - Recall the definition of continuity:
    * Let $(X,\, d_X)$ and $(Y,\, d_Y)$ be two metric spaces and let $f : X \to Y$.
    * f is continuous at $x_0 \in X$ iff $\forall \varepsilon > 0\ \exists \delta(\varepsilon,\, x_0) > 0$ such that $d_X(x_0,\, x) < \delta(\varepsilon,\, x_0) \implies d_Y(f(x_0),\, f(x)) < \varepsilon$ for all $x \in X$.
    * Then if f is continuous at $x_0 \in X$ for *all* $x_0 \in X$ then f is continuous.
  - If $a = \lim_{n\to\infty} a_n$ exists, then it is generally true that $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$ whenever $f : \mathbb{R} \to \mathbb{R}$ is a continuous function:
    Proof:
    * We want to show that $\lim_{n\to\infty} f(a_n) = f(a)$.
    * But $(f(a_n))_{n\in\mathbb{N}}$ defines a sequence in itself, and so what we need is to show that $(f(a_n))_{n\in\mathbb{N}} \to f(a)$.
    * To that end, let $\varepsilon > 0$ be given. We need to find some $m(\varepsilon) \in \mathbb{N}$ such that if $n > m(\varepsilon)$ then $|f(a_n) - f(a)| < \varepsilon$.
    * However, we know that f is continuous at a, so that $\exists \delta(\varepsilon,\, a) > 0$ such that if $|a_n - a| < \delta(\varepsilon,\, a)$ then indeed we will have $|f(a_n) - f(a)| < \varepsilon$. But $a_n \to a$, which means that $\exists l(\delta(\varepsilon,\, a)) \in \mathbb{N}$ such that if $n \geq l$ then $|a_n - a| < \delta(\varepsilon,\, a)$.
    * So take $m(\varepsilon) := l(\delta(\varepsilon,\, a))$.
  - So assume that u and v are fixed and define a funtion $f : \mathbb{R} \to \mathbb{R}$ by $x \overset{f}{\mapsto} \langle xu,\, v\rangle$. We want to show that f is continuous.
  - Because for our f, the two metric spaces are just $\mathbb{R}$ with the Euclidean metric, we have to show that:
    * Let $x_0 \in \mathbb{R}$ be given.
    * $\forall \varepsilon > 0\ \exists \delta(\varepsilon,\, x_0) > 0$ such that if $x \in \mathbb{R}$ is such that $|x - x_0| < \delta(\varepsilon)$ then $|f(x_0) - f(x)| < \varepsilon$. For our f $|f(x_0) - f(x)| < \varepsilon$ would mean $|\langle x_0 u,\, v\rangle - \langle xu,\, v\rangle| < \varepsilon$.

  * Estimate the following:

$$
\begin{aligned}
\left|\langle x_0 u,\, v\rangle - \langle xu,\, v\rangle\right| &= \left|\langle x_0 u - xu,\, v\rangle\right| \\
&= \left|\langle (x_0 - x)\, u,\, v\rangle\right| \\
&\leq \|(x_0 - x)\, u\|\|v\| \\
&= |x_0 - x|\, \|u\|\|v\|
\end{aligned}
$$

  * So let $\varepsilon > 0$ be given. Pick $\delta(\varepsilon) = \frac{\varepsilon}{\|u\|\|v\|}$.
  * Then if $|x - x_0| \leq \frac{\varepsilon}{\|u\|\|v\|}$, then $|\langle x_0 u,\, v\rangle - \langle xu,\, v\rangle| \leq \frac{\varepsilon}{\|u\|\|v\|}\|u\|\|v\| = \varepsilon$ and so f is continuous.

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1.2.2 Question 4

- Pick the axes so that $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Can always do that.
- Then we know that the distance of a from E is just the z-component of a: $a \cdot \hat{z}$. Thus we are finished, we just need to *prove* the result.
- To prove it, define $x_0 := a - a \cdot \hat{z}$. It is clear that x_0 lies in the $x - y$-plane (E) because we removed from a the z-component.
- Write $a \cdot \hat{z} = a - x_0$.
- Now we want to show that $|a - x_0| = \inf(\{\, |a - x| \mid x \in E \,\})$.
- To that end, we need to show that $|a - x_0|$ is (1) a lower bound, and that it is (2) the highest lower bound.
  1. Let $x \in E$ be given.
  2. Then $|a - x|^2 = |a - x_0|^2 + |x - x_0|^2 \geq |a - x_0|^2$. Thus $|a - x_0|$ is a lower bound.
  3. Because $x_0 \in E$ then this lower bound is actually a minimum and as such it is also the infimum (whenever there is a minimum, the infimum is equal to it).

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1.2.3 Question 6

- *Claim:* $\lim_{n\to\infty} \sqrt{n}\,(\sqrt[n]{n} - 1) = 0$.
  Proof:
  - Define $x_n := \sqrt[n]{n} - 1$. Then $(x_n + 1)^n = n$.
  - But for all $n \geq 4$ we have

$$
\begin{aligned}
n = (x_n + 1)^n &= \sum_{j=0}^{n} \binom{n}{j} (x_n)^j = \binom{n}{3} (x_n)^3 + \sum_{j \in \{0,\, \dots,\, n\} \setminus \{3\}} \binom{n}{j} (x_n)^j \\
&\geq \binom{n}{3} (x_n)^3 \\
&= \frac{n\,(n-1)\,(n-2)\,(n-3)\dots 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot (n-3)\dots 3\cdot 2\cdot 1} (x_n)^3 \\
&= \frac{n\,(n-1)\,(n-2)}{6} (x_n)^3 \\
&\overset{*}{\geq} \frac{n\cdot n\,(n-3)}{6} (x_n)^3 \\
&= \frac{n^3}{6}\frac{n-3}{n} (x_n)^3 \\
&\overset{**}{\geq} \frac{n^3}{6}\frac{1}{4} (x_n)^3 \\
&= \frac{n^3}{24} (x_n)^3
\end{aligned}
$$

  where in $*$ we have used that $(n-1)\,(n-2) \geq n\,(n-3)$ for all $n \geq 4$ and in $**$ we have used that $\frac{n-3}{n} \geq \frac{1}{4}$ for all $n \geq 4$ (both facts which you should prove with induction if you don't agree with them).
  - Thus we have $x_n \leq \sqrt[3]{24 n^{1-3}} = \sqrt[3]{24} n^{-\frac{2}{3}}$.

- Thus we have $\sqrt{n}x_n \le \sqrt[3]{24}n^{-\frac{2}{3}+\frac{1}{2}} = \sqrt[3]{24}n^{-\frac{1}{6}}$.
- But $\sqrt{n}x_n \ge 0$. So taking the lim sup and lim inf of both sides, using the fact that $\lim_{n\to\infty} n^{-\frac{1}{6}} = 0$ we get the desired result.

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2 Exercise Sheet Number 7

2.1 Absolutely Converging Series

- $\sum a_n$ converges absolutely if $\sum |a_n|$ converges absolutely.
- For a series of positive terms of course there is no difference between the two notions.
- Clearly if $\sum a_n$ converges absolutely then $\sum a_n$ converges.
  - The converse is false, for example, take $\sum \frac{(-1)^n}{n}$ which converges, but not absolutely.
- The cool thing about series which converge absolutely is that we may rearrange the sum in any way we like and we'd still get the same converging result:
  - A rearrangement of a series $\left(\sum_{j=1}^n a_j\right)_{n\in\mathbb{N}}$ is a new series specified by the bijection $f : \mathbb{N} \to \mathbb{N}$ as $\left(\sum_{j=1}^n a_{f(j)}\right)_{n\in\mathbb{N}}$.
  - For example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$ converges to something (call it s). However, $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$ converges also, but to something else.
  - However, if $\sum a_n$ converges absolutely, then *any rearrangement of* $\sum a_n$ *converges to the same sum.* (A fact I hope was proven in class).
- You will have to use this fact extensively with the zeta function.

2.2 Power Series

- A power series is an infinite series of the form $\left(\sum_{j=1}^n a_j z^j\right)_{n\in\mathbb{N}}$ where $(a_j)_{j\in\mathbb{N}}$ is some sequence and $z \in \mathbb{C}$ is some complex number. Thus, if the series converges depends on the value of z, and of course also the sum to which it converges.
- We can find the "radius of convergence" by applying the root test:
  - $\sum f_n$ converges if $\limsup_{n\to\infty} \sqrt[n]{|f_n|} < 1$ and diverges if $\limsup_{n\to\infty} \sqrt[n]{|f_n|} > 1$. If $\limsup_{n\to\infty} \sqrt[n]{|f_n|} = 1$ the test gives no information.
  - Thus for our power series that would be $\limsup_{j\to\infty} \sqrt[j]{|a_j z^j|} = |z| \limsup_{j\to\infty} \left(\sqrt[j]{|a_j|}\right) \overset{!}{<} 1$ that is, $|z| < \frac{1}{\limsup_{j\to\infty}\left(\sqrt[j]{|a_j|}\right)}$ and so the radius of convergence is $\frac{1}{\limsup_{j\to\infty}\left(\sqrt[j]{|a_j|}\right)}$.
  - The ratio test doesn't say what happens *on* the radius!
- The convergence radius guarantees *absolute* convergence. Thus, inside the radius of convergence, we may rearrange the summation.

2.3 Concrete Tips for the Questions

2.3.1 Question 1

- For part (a): The series (the zeta function series) converges absolutely. That means we can sum term by term. add and remove $\sum \frac{1}{(2n)^2}$ from the left hand side.
- For part (b): Use fraction decomposition: $\frac{1}{n(n+1)(n+2)} = \frac{p_1}{n} + \frac{p_2}{n+1} + \frac{p_3}{n+1}$. Find p_1, p_2, and p_3.
- For part (c): Prove $\frac{1}{f_n f_{n+2}} = \frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1} f_{n+2}}$. Then use homework 1 exercise 6 (b).

2.3.2 Question 2

- For part (a): Use the result for $\frac{f_{n+1}}{f_n}$ from homework 6, exercise 3 (c). Then use again the fact that for a power series, convergence in the radius of convergence is absolute to conclude that you may sum up terms one by one. Then compute $\left(1 - z - z^2\right) f(z)$.
- For part (b): Plug in the expression given for f_n into $\sum_{n=0}^{\infty} f_n z^n$, use absolute convergence to sum term by term, and then use the geometric series formula.

2.3.3 Question 3

- Part (a): no hint as this should take you 3 lines (no induction please!).
- Part (b): Find the radius of convergence. Then the hard part is not what happens inside the radius or outside of it, but rather what happens *on* the radius. Show that except for one point on the radius, all other points make the series converge. To that end, use part (a) with $b_k = \frac{1}{k}$ and $a_k = \frac{1-z^k}{1-z}$. You will be able to decompose the series and show that each constituent converges.

2.3.4 Question 4

- Prove $\exp(x) \geq 1 + x$ for all $x \geq 0$.
- Prove $\exp(x)\exp(y) = \exp(x+y)$ for all $(x,\, y) \in \mathbb{C}^2$.
- Prove that exp is a monotone increasing map.
- Define $s_n := \sum_{j=1}^{n} a_j$ and $p_n := \prod_{j=1}^{n}(1+a_j)$.
- Show that $\lim_{n\to\infty} s_n$ exists $\iff$ $\lim_{n\to\infty} p_n$ exists.
- $\implies$ is shown using the properties of the exp map.
- $\impliedby$ is shown by "multiplying out" $\prod_{j=1}^{n}(1+a_j)$ to show it's bigger than $\sum_{j=1}^{n} a_j$. Think what happens for the first few cases: $(1+a_1)(1+a_2) =?$ and so on and from there you'll hopefully see a pattern.

2.3.5 Question 5

- $J_N \equiv \left\{ \prod_{j=1}^{N} (p_j)^{\alpha_j} \,\middle|\, a_j \in \mathbb{N} \cup \{0\}\, \forall j \in \{1,\, \ldots,\, N\} \right\}$.
- Use absolute convergence of the zeta function.
- Prove that $\sum_{(a_1,\, \ldots,\, \alpha_N) \in (\mathbb{N}\cup\{0\})^N} \left(\prod_{k=1}^{N} (f_k)^{\alpha_k} \right) = \prod_{k=1}^{N} \left(\sum_{\alpha_k=1}^{N} (f_k)^{\alpha_k} \right)$ for any sequence f_k.
- Then use the geometric series.

2.3.6 Question 6

- Part (a): Use the definition of continuity.