Analysis 1
Recitation Session of Week 8

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1 Exercise Sheet Number Six

1.1 Question 1

- Cannot do $\lim \frac{\sqrt{1+a_n}-1}{a_n} \stackrel{?}{=} \frac{1}{2} \Longleftrightarrow \lim \sqrt{1+a_n} - 1 = \lim \frac{a_n}{2}$.

1.2 Question 2

- Let $(a_n)_{n\in\mathbb{N}}$ converge to a.
- Define $s_n := \frac{1}{n}\sum_{j=1}^n a_j$ for all $n \in \mathbb{N}$.
- *Claim*: $(s_n)_{n\in\mathbb{N}}$ converges to a.
  Proof:
  - Let $\varepsilon_0 > 0$ be given.
  - $(a_n)_{n\in\mathbb{N}} \to a$ means that $\forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N}$ such that if $n \geq m(\varepsilon)$ then $|a - a_n| < \varepsilon$.
  - Then assume that $n \in \mathbb{N}$ is such that $n > m\left(\frac{\varepsilon_0}{2}\right)$, for some $\varepsilon > 0$. Then,

$$\begin{aligned}
|s_n - a| &= \left|\frac{1}{n}\sum_{j=1}^n a_n - a\right| \\
&= \left|\frac{1}{n}\sum_{j=1}^n a_n - \underbrace{\frac{1}{n}\sum_{j=1}^n}_{1} a\right| \\
&= \left|\frac{1}{n}\sum_{j=1}^n (a_n - a)\right| \\
&= \left|\frac{1}{n}\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)} (a_n - a) + \frac{1}{n}\sum_{j=m\left(\frac{\varepsilon_0}{2}\right)+1}^{n} (a_n - a)\right| \\
&\leq \frac{1}{n}\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)} (a_n - a)\right| + \frac{1}{n}\sum_{j=m\left(\frac{\varepsilon_0}{2}\right)+1}^{n} |(a_n - a)| \\
&\leq \frac{1}{n}\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)} (a_n - a)\right| + \frac{1}{n}\sum_{j=m\left(\frac{\varepsilon_0}{2}\right)+1}^{n} \frac{\varepsilon_0}{2} \\
&= \frac{1}{n}\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)} (a_n - a)\right| + \underbrace{\frac{n - m\left(\frac{\varepsilon_0}{2}\right) - 1}{n}}_{\leq 1}\frac{\varepsilon_0}{2} \\
&\leq \frac{1}{n}\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)} (a_n - a)\right| + \frac{\varepsilon_0}{2}
\end{aligned}$$

  - Then, can we pick some $n \in \mathbb{N}$ large enough such that $\frac{1}{n}\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)} (a_n - a)\right| + \frac{\varepsilon_0}{2} < \varepsilon_0$?

– Sure, define $m_s(\varepsilon_0) := 1 + 2\frac{\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)}(a_n - a)\right|}{\varepsilon_0}$. Note that $\left|\sum_{j=1}^{m\left(\frac{\varepsilon_0}{2}\right)}(a_n - a)\right|$ is just some finite fixed number.
– Then clearly if $n > m_s(\varepsilon_0)$ then $|s_n - a| < \varepsilon_0$.

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- For part (b) consider $a_n = (-1)^n$ which diverges.

1.3 Question 3

- Merely Positive or merely monotone sequence is not enough in order for it to converge. Needs to be bounded as well.
- Counter examples:
  – $\{1, 2, 1, 2, 1, 2, \dots\}$ is always positive but surely diverges.
  – $\{1, 2, 3, 4, \dots\}$ is monotone increasing but diverges.
- If we have a monotone increasing sequence that is bounded above then it will converge.
- If we have a monotone decreasing sequence that is bounded below (for example, always positive) then it will converge.

1.4 Question 4

- Let $p \in \mathbb{P}$ be given. Define $d_p : \mathbb{Z}^2 \to \mathbb{R}$ by: $d_p((m, n)) := \begin{cases} 0 & m = n \\ \min\left(\left\{ p^{-k} \in \mathbb{Q} \,\middle|\, k \in \mathbb{N} \wedge p^k | (m-n) \right\}\right) & m \neq n \end{cases}$. The minimum is taken over the set of all p^{-k} where k ranges over all natural numbers which satisfy $p^k | (m - n)$.
- *Claim*: $d_p(m, n) \leq d_p(m, l) + d_p(l, n)$ for all $l \in \mathbb{Z}$.
  Proof:
  – *Case 1*: $p = 1$.
    * When $p = 1$, then every $p^k = 1$ for all k. As a result, $d_1(m, n) = 1$ for all $m \neq n$.
    * Then we have $d_1(m, n) = 1$ and indeed $1 \leq 1 + 0$ or $1 \leq 0 + 1$.
  – *Case* 2: $p > 1$.
    * Let k_1 be the largest power of p inside of $m - l$: $m - l = \alpha p^{k_1}$ and $d_p(m, l) = p^{-k_1}$.
    * Let k_2 be the largest power of p inside of $l - n$: $l - n = \beta p^{k_2}$ and $d_p(l, n) = p^{-k_2}$.
    * Define $k_0 := \min(k_1, k_2)$.
    * Then

$$\begin{aligned} m - n &= m - l + l - n \\ &= (m - l) + (l - n) \\ &= \alpha p^{k_1} + \beta p^{k_2} \\ &= p^{k_0}\left(\alpha p^{k_1 - k_0} + \beta p^{k_2 - k_0}\right) \end{aligned}$$

    * As a result, $p^{k_0} | (m - n)$ and so $p^{-k_0} \geq d_p(m, n)$ (by definition of d_p).
    * But becuase $k_0 \equiv \min(k_1, k_2)$, $k_0 = k_1$ or $k_0 = k_2$, and of course $p^{-k_0} \leq p^{-k_0} +$ something.
    * Hence $p^{-k_0} \leq p^{-k_1} + p^{-k_2}$ necessarily.
    * Hence our result follows.

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1.5 Question 5

- Let (X, d) be a metric space, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and let $x \in X$.
- Part (a): *Claim*: If $(x_n)_{n \in \mathbb{N}} \to x$ then $(x_n)_{n \in A} \to x$ for all $A \subseteq \mathbb{N}$ such that $|A| = |\mathbb{N}|$.
  Proof:
  – Because $(x_n)_{n \in \mathbb{N}} \to x$, then $\forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N}$ such that if $n \in \mathbb{N}$ is such that $n \geq m(\varepsilon)$ then $d(x_n, x) < \varepsilon$.
  – To show that $(x_n)_{n \in A} \to x$, we need to show that $\forall \varepsilon > 0 \exists m_A(\varepsilon) \in A$ such that if $n \in A$ is such that $n \geq m_A(\varepsilon)$ then $d(x_n, x) < \varepsilon$.
  – So let $\varepsilon > 0$ be given.
  – If $m(\varepsilon)$ happens to be such that $m(\varepsilon) \in A$, define $m_A(\varepsilon) := m(\varepsilon)$.

- Otherwise, because $|A| = |\mathbb{N}|$, there must be some member of A, a such that $a > m(\varepsilon)$. So define $m_A(\varepsilon) := a$.
- Then we are done, because if $n \in A$ such that $n \geq m_A(\varepsilon)$, then
  * Due to $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$.
  * Due to $m_A(\varepsilon) \geq m(\varepsilon)$, $n \geq m(\varepsilon)$.
  * Thus due to $(x_n)_{n\in\mathbb{N}} \to x$ we have that $d(x_n, x) < \varepsilon$.

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- Part (b): *Claim*: If for every $A \subseteq \mathbb{N}$ such that $|A| = |\mathbb{N}|$, $\exists B \subseteq A$ such that $|B| = |\mathbb{N}|$ such that $(x_n)_{n\in B} \to x$, then $(x_n)_{n\in\mathbb{N}} \to x$.
  Proof:
  - Assume the contrary, that is, assume that $(x_n)_{n\in\mathbb{N}}$ does *not* converge to x.
  - That means that $\exists \varepsilon_0 > 0$ such that $\forall m \in \mathbb{N}$, $\exists n_0(m) \in \mathbb{N}$ such that $n_0(m) > m$ yet $d\left(x_{n_0(m)}, x\right) \geq \varepsilon_0$.
  - Define a subset $A \subseteq \mathbb{N}$ such that $|A| = |\mathbb{N}|$ by the following rule:
    * Define $a_1 := n_0(1)$.
    * Define $a_2 := n_0(2)$.
    * etc.
    * Then $A := \{ a_j \mid j \in \mathbb{N} \}$.
  - But then we have a contradiction with the fact that *any* subsequence B of A converges, because we can simply take $B = A$, and clearly, that subsequence $(x_n)_{n\in A}$ does not converge.

1.6 Question 6

- If $(a_n)_{n\in\mathbb{N}} \to 0$ then it doesn't necessarily mean that $(\sum a_n)$ converges!
- You may not manipualte limits as if they were numbers before you know that they actually converge!

2 Exercise Sheet Number Eight

2.1 Continuity of Complex Functions

- *Claim*: $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $z \overset{f}{\mapsto} \frac{z^2}{|z|}$ is continuous.
  Proof:
  - *Claim*: If $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous then so is $g \circ f : X \to Z$.
    Proof:
    * Let $\varepsilon > 0$ be given, and pick some $x_0 \in X$. Then $f(x_0) \in Y$.
    * Because g is continuous (in particular, continuous at $f(x_0)$), $\exists \delta_Y(\varepsilon, f(x_0)) > 0$ such that for all $y \in Y$ with $d_Y(y, f(x_0)) < \delta_Y(\varepsilon, f(x_0))$ we have $d_Z(g(y), g \circ f(x_0)) < \varepsilon$.
    * Because f is continuous (in particular, continuous at x_0) $\exists \delta_X(\varepsilon, x_0) > 0$ such that for all $x \in X$ with $d_X(x, x_0) < \delta_X(\varepsilon, x_0)$ we have $d_Y(f(x), f(x_0)) < \varepsilon$.
    * Apply continuity of f on the radius $\delta_Y(\varepsilon, f(x_0))$ at x_0: there exists some $\delta_X(\delta_Y(\varepsilon, f(x_0)), x_0) > 0$ such that:
      · If $x \in X$ is such that $d_X(x, x_0) < \delta_X(\delta_Y(\varepsilon, f(x_0)), x_0)$ then $d_Y(f(x), f(x_0)) < \delta_Y(\varepsilon, f(x_0))$.
    * But $f(x) \in Y$ such that $d_Y(f(x), f(x_0)) < \delta_Y(\varepsilon, f(x_0))$ implies that $d_Z(g(f(x)), g \circ f(x_0)) < \varepsilon$.

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  - *Claim*: Let $g : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ be continuous. Then the map $h : \mathbb{C} \to \mathbb{C}$ defined by $q(z) := \frac{1}{g(z)}$ for all $z \in \mathbb{C}$ is continuous.
    Proof:
    * Let $\varepsilon_0 > 0$ be given and take some $z_0 \in \mathbb{C}$.
    * Compute

$$\begin{aligned} |q(z_0) - q(z)| &\equiv \left|\frac{1}{g(z_0)} - \frac{1}{g(z)}\right| \\ &= \left|\frac{g(z) - g(z_0)}{g(z_0)\, g(z)}\right| \end{aligned}$$

    * Because g is continuous at $g(z_0)$ then $\forall \varepsilon > 0$ $\exists \delta(\varepsilon, z_0) > 0$ such that if $|z - z_0| < \delta(\varepsilon, z_0)$ then $|g(z) - g(z_0)| < \varepsilon$.
    * Using this last inequality we can also infer that $|g(z_0)| - |g(z)| < \varepsilon$ and so $|g(z)| > |g(z_0)| - \varepsilon$.
    * Assume $|g(z_0)| \neq \varepsilon$ (otherwise pick ε slightly smaller for the same z_0).
    * Then we have $\frac{1}{|g(z)|} < \frac{1}{|g(z_0)| - \varepsilon}$.

* As a result we find that

$$|q(z_0) - q(z)| \leq \frac{\varepsilon}{|g(z_0)|\,[|g(z_0)| - \varepsilon]}$$

* So take $\delta\left(\frac{|g(z_0)|^2\varepsilon_0}{1+|g(z_0)|\varepsilon_0},\, z_0\right)$.
* Then

$$\begin{aligned}|q(z_0) - q(z)| &\leq \frac{\frac{|g(z_0)|^2\varepsilon_0}{1+|g(z_0)|\varepsilon_0}}{|g(z_0)|\left[|g(z_0)| - \frac{|g(z_0)|^2\varepsilon_0}{1+|g(z_0)|\varepsilon_0}\right]} \\ &= \frac{\frac{|g(z_0)|^2\varepsilon_0}{1+|g(z_0)|\varepsilon_0}}{\frac{|g(\varepsilon_0)|^2}{1+|g(z_0)|\varepsilon_0}} \\ &= \varepsilon_0\end{aligned}$$

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- *Claim*: If $f:\mathbb{C}\to\mathbb{C}$ and $g:\mathbb{C}\to\mathbb{C}$ are continuous then $h:\mathbb{C}\to\mathbb{C}$ defined as $h(z) := f(z)\,g(z)$ for all $z\in\mathbb{C}$ is continuous.
  Proof:
  * Let $\varepsilon_0 > 0$ be given and let $z_0 \in \mathbb{C}$ be given.
  * f is continuous so $\exists\delta_f(\varepsilon, z_0) > 0$ such that $|z - z_0| < \delta_f(\varepsilon, z_0)$ leads to $|f - f(z_0)| < \varepsilon$.
  * Same for g, denoted by $\delta_g(\varepsilon, z_0)$.
  * Then

$$\begin{aligned}|h(z) - h(z_0)| &= |f(z)\,g(z) - f(z_0)\,g(z_0)| \\ &= |f(z)\,g(z) - g(z_0)\,f(z_0) + g(z_0)\,f(z_0) - f(z_0)\,g(z_0)| \\ &= |f(z)\,[g(z) - g(z_0)] + g(z_0)\,[f(z) - f(z_0)]| \\ &\leq |f(z)|\,|g(z) - g(z_0)| + |g(z_0)|\,|f(z) - f(z_0)|\end{aligned}$$

  * Using the fact that f is continuous, we have $|f(z)| < \varepsilon + |f(z_0)|$ so some suitable selection of z.
  * As a result we find that

$$\begin{aligned}|h(z) - h(z_0)| &\leq [\varepsilon + |f(z_0)|]\,|g(z) - g(z_0)| + |g(z_0)|\,|f(z) - f(z_0)| \\ &\leq [\varepsilon + |f(z_0)|]\,\varepsilon + |g(z_0)|\,\varepsilon \\ &= \varepsilon^2 + [|f(z_0) + g(z_0)|]\,\varepsilon\end{aligned}$$

  * So take

$$\delta(\varepsilon_0, z_0) := \min\left(\left\{\begin{array}{c}\delta_f\left(\frac{1}{2}\left(-[|f(z_0)| + |g(z_0)|] + \sqrt{[|f(z_0)| + |g(z_0)|]^2 + 4\varepsilon_0}\right),\, z_0\right), \\ \delta_g\left(\frac{1}{2}\left(-[|f(z_0)| + |g(z_0)|] + \sqrt{[|f(z_0)| + |g(z_0)|]^2 + 4\varepsilon_0}\right),\, z_0,\, z_0\right)\end{array}\right\}\right)$$

  * Then we have $|h(z) - h(z_0)| \leq \varepsilon_0$.

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- Then clearly $z \mapsto z^2$ which is just the multiplication of two identity maps is continuous.
- *Claim*: $||:\mathbb{C}\to\mathbb{R}$ defined by $z\mapsto |z|$ is continuous.
  Proof:
  * Let $\varepsilon > 0$ be given and let $z_0 \in \mathbb{C}$ be given.
  * Then we need

$$||z| - |z_0|| \leq |z - z_0|$$

  * So take $\delta(\varepsilon, z_0) := \varepsilon$.
  * Then if $|z - z_0| < \delta(\varepsilon, z_0)$ then $||z| - |z_0|| < \varepsilon$ and we're done.

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- Putting everything together, we have the following maps:
  * $f_1:\mathbb{C}\to\mathbb{C}$ defined by $z\mapsto z^2$. This map is continuous.
  * $f_2:\mathbb{C}\to\mathbb{C}$ defined by $z\mapsto |z|$. This map is continuous.
  * $f_3:\mathbb{C}\backslash\{0\}\to\mathbb{C}$ defined by $z\mapsto \frac{1}{z}$. This map is continuous.
  * Then $f = f_1\cdot(f_3\circ f_2)$. Since all the operations were proven to be continuous we have proven that our map is continuous.

2.2 Continuos Extensions

- *Claim*: $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ defined by $x\overset{f}{\mapsto}\frac{\sin(x)}{x}$ is continuous.
  Proof:
  - Even though the trigonometric functions have not officially been defined yet, we can think of them as being defined as a power series.
  - For example, define $\exp:\mathbb{C}\to\mathbb{C}$ as:
  $$\exp(z)\equiv\sum_{n=0}^{\infty}\frac{z^n}{n!}$$
  Using the ratio test, we have that
  $$\left|\frac{\left[\frac{z^{n+1}}{(n+1)!}\right]}{\left(\frac{z^n}{n!}\right)}\right|=\left|\frac{z}{n+1}\right|$$
  and so $\limsup_{n\to\infty}\left|\frac{\left[\frac{z^{n+1}}{(n+1)!}\right]}{\left(\frac{z^n}{n!}\right)}\right|=\limsup_{n\to\infty}\left|\frac{z}{n+1}\right|=|z|\cdot 0<1$.
  - As a result, $\exp(z)$ converges and so is well-defined.
  - It is continuous on any bounded subset of $\mathbb{C}$ because using the Weierstrass M-test with $\frac{R^n}{n!}$ (where R is the radius of the bounded set), we have uniform convergence. As each element $f_n(z)\equiv\frac{z^n}{n!}$ is continuous, $\exp(z)$ is continuous as well (there's a more rigorous way to show continuity).
  - It may seem crazy but $\sin(z)\equiv\frac{1}{2i}[\exp(iz)-\exp(-iz)]$. Due to the continuity of exp and the theorems above we have that sin is also continuous. One can show that if $z\in\mathbb{R}$ then $\sin(z)\in\mathbb{R}$ as well and so our initial function is well defined.
  - Of course $x\mapsto x$ is also continuous.

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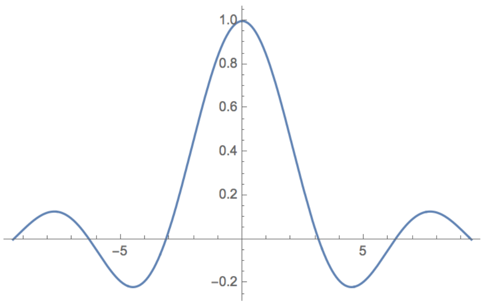
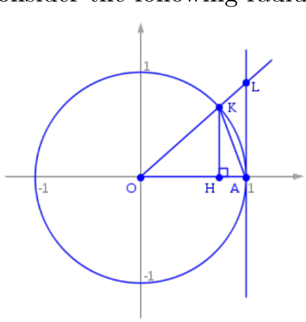
- Of course, f is not defined at 0.
- Here's a picture of f none the less:



- *Claim*: $\lim_{x\to 0}\frac{\sin(x)}{x}=1$.
  Proof:
  - *Claim*: We can use the fact that $\cos(x)\le\frac{\sin(x)}{x}\le 1$.
    Proof: (proof using trigonometry).
    * Consider the following radius-1 circle:



    * Let the angle HOK be denoted by x.
    * The area of the triangle ΔKOA is equal to $\frac{\sin(x)\cdot 1}{2}$.
    * The area of the sector KOA is given by $\frac{x}{2\pi\cdot 1}\pi(1)^2=\frac{x}{2}$.
    * The area of the triangle ΔLOA is: $\frac{LA}{OA}=\tan(x)$ so that $LA=\tan(x)$. Then the area is $\frac{1}{2}LA=\frac{1}{2}\tan(x)$.
    * But due to the different areas containing each other we have $\frac{1}{2}\sin(x)\le\frac{1}{2}x\le\frac{1}{2}\tan(x)$.

        * Thus $1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$ and so $\cos(x) \le \frac{\sin(x)}{x} \le 1$.
    - Now use the the fact that if $a_n \le b_n$ then $\limsup a_n \le \limsup b_n$ and same for $\liminf$, and the fact that $\cos(0) = 1$ and that cos is continuous at 0.

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- As a result it makes sense to define f at 0 to be 1.
- We have just shown that by employing this definition we make sin continuous at 0.

2.3 Concrete Tips for Questions

- Question 2:
    - Show uniform convergence of the series of functions $f_n(z) \equiv \frac{2z}{z^2-n^2}$.
    - You will not be able to show this for all $\mathbb{C}\backslash\mathbb{Z}$. Show it only for some bounded area of $\mathbb{C}$.
    - Use the Weierstrass M-test we discussed in the last colloquium with a series built on top of the zeta-function.
    - For periodicity use $\frac{2z}{z^2-n^2} = \frac{1}{z+n} + \frac{1}{z-n}$.
- Question 3:
    - For (a):
        * Again the M-test with $M_n = s_n$.
    - For (b):
        * Show that $f({a_n}^+) - f(a_n) = s_n = f(a_n) - f({a_n}^-)$.
- Question 4:
    - Define $h(x) = f(x) - f\left(x + \frac{1}{2}\right)$. See what you get.
- Question 5:
    - For part (a):
        * One direction of the proof is trivial. Which is it?
        * The other direction:
            · Use the fact that a finite union of closed sets is again closed.
- Question 6:
    - We have done this in the colloquium on the Cantor set (except for the continuity property, which you can find in the corresponding exercise in Koenigsberger).