# Analysis 1
# Recitation Session of Week 9

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## 1 Exercise Sheet Number 7

### 1.1 Question 1

- How to use absolute convergence and what for: *where* exactly does the rearrangement take place?
- In $(b)$, why do we need to work with partial sums first? Because $\sum \frac{1}{n}$ doesn't converge.
- Part (a): *Claim*: $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3}{4}\zeta(2)$.
  *Proof*:
  - Note: I prefer the proofs that start from one side of the equation and then the other side of the equation. It feels more direct to me than somehow manipulating the equation to show in the end something like $1=1$. In this spirit:
  - Add and substract $\frac{1}{4}\zeta(2)$ from $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\zeta(2) - \frac{1}{4}\zeta(2) \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4}\zeta(2) \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} - \frac{1}{4}\zeta(2) \\
&= \left[\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{(2n-1)^2}\right] + \left[\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{(2n)^2}\right] - \frac{1}{4}\zeta(2) \\
&\overset{(1)}{=} \lim_{N\to\infty} \left[\left(\sum_{n=1}^{N} \frac{1}{(2n-1)^2}\right) + \left(\sum_{n=1}^{N} \frac{1}{(2n)^2}\right)\right] - \frac{1}{4}\zeta(2) \\
&\overset{(2)}{=} \lim_{N\to\infty} \left[\sum_{n=1}^{N} \left(\frac{1}{(2n-1)^2} + \frac{1}{(2n)^2}\right)\right] - \frac{1}{4}\zeta(2) \\
&= \lim_{N\to\infty} \left[\sum_{n=1}^{2N} \frac{1}{n^2}\right] - \frac{1}{4}\zeta(2) \\
&= \frac{3}{4}\zeta(2)
\end{aligned}$$

  - Note that at (1) we have used the fact that if $\lim a_n$ exists and $\lim b_n$ exists then $\lim(a_n+b_n)$ exists as well and is equall to $\lim(a_n+b_n) = (\lim a_n) + (\lim b_n)$.
  - Note that at (2) we have used the property of absolutely convergent series ($\lim_{N\to\infty}\left[\sum_{n=1}^{2N} \frac{1}{n^2}\right]$ converges absolutely) which allowed us to rearrange the summation from $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots$ into $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\dots$.
  - Many people did not give the justification for these two transitions, taking them for granted. To fully solve this question, you *must prove the absolute convergence of* $\zeta(2)$.

  $\blacksquare$

- For (b): *Claim*: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.
  *Proof*:

- Write

$$
\begin{aligned}
\frac{1}{n(n+1)(n+2)} &= \frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2(n+2)}\\
&= \frac{1}{2}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)\\
&= \frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right)\\
&= -\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+2}-\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)
\end{aligned}
$$

- Then use a result from HW1 saying that $\sum_{n=1}^{N}(a_n-a_{n-1})=a_N-a_0$ by defining $a_n:=\frac{1}{n+1}-\frac{1}{n+2}$ and then we have

$$
\begin{aligned}
\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} &= -\frac{1}{2}\sum_{n=1}^{N}(a_n-a_{n-1})\\
&= -\frac{1}{2}(a_N-a_0)\\
&= -\frac{1}{2}\left(\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{1}+\frac{1}{2}\right)\\
&= -\frac{1}{2}\left(\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{2}\right)
\end{aligned}
$$

- Now clearly $\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)}=\lim -\frac{1}{2}\left(\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{2}\right)=\frac{1}{4}$.
- Some people wrote crazy things like $\sum_{n=1}^{\infty}\frac{1}{n+1}-\sum_{n=1}^{\infty}\frac{1}{n+1}$ and cancelled it out as zero. This is *false* because these things do not converge! You must work with partial sums first and only in the end take the limit to zero.

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## 1.2 Question 3

- We have the series $\sum_{k=1}^{\infty}\frac{z^k}{k}$.
- Using the ratio test we have that this series converges if

$$\limsup_{k\to\infty}\left|\frac{\frac{z^{k+1}}{k+1}}{\frac{z^k}{k}}\right|<1$$

or if

$$\underbrace{\limsup_{k\to\infty}\left|\frac{k}{k+1}\right|}_{1}<\frac{1}{|z|}$$

thus the radius of convergence is 1.

- Thus we have that $\sum_{k=1}^{\infty}\frac{z^k}{k}$ converges if $|z|<1$ and diverges if $|z|>1$. There is no more information from the root test about what happens at $|z|=1$.
- For $z=1$ we know that $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.
- *Claim*: $\sum_{k=1}^{\infty}\frac{z^k}{k}$ converges for all $z\in\{\,w\in\mathbb{C}\mid |w|=1\wedge w\neq 1\,\}$.
*Proof*:
  - For $z=-1$ we know that the series converges because we have the alternating series: $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$.
  - Otherwise, define $b_k:=\frac{1}{k}$ and $a_k:=\sum_{j=0}^{k-1}z^j$ (recall that $\sum_{j=0}^{k-1}z^j=\frac{1-z^k}{1-z}$).
  - Observe that

$$
\begin{aligned}
a_{k+1}-a_k &= \sum_{j=0}^{k}z^j-\sum_{j=0}^{k-1}z^j\\
&= z^k
\end{aligned}
$$

– Then we have that

$$\begin{aligned}\sum_{k=1}^{n}\frac{z^k}{k} &= \sum_{k=1}^{n} b_k\left(a_{k+1}-a_k\right)\\ &\overset{(a)}{=} a_{n+1}b_{n+1}-a_1b_1-\sum_{k=1}^{n}a_{k+1}\left(b_{k+1}-b_k\right)\\ &= \left(\underbrace{\sum_{j=0}^{n}z^j}_{\frac{1-z^{n+1}}{1-z}}\right)\left(\frac{1}{n+1}\right)-\left(\underbrace{\sum_{j=0}^{0}z^j}_{1}\right)\frac{1}{1}-\sum_{k=1}^{n}\left[\left(\underbrace{\sum_{j=0}^{k}z^j}_{\frac{1-z^{k+1}}{1-z}}\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right]\\ &= \left(\frac{1-z^{n+1}}{1-z}\right)\left(\frac{1}{n+1}\right)-1-\sum_{k=1}^{n}\left[\left(\frac{1-z^{k+1}}{1-z}\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right]\end{aligned}$$

– Now we can show that

1. $\lim_{n\to\infty}\left(\frac{1-z^{n+1}}{1-z}\right)\left(\frac{1}{n+1}\right)=0$ (using the fact that $|z|=1$).
2. $\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(\frac{1-z^{k+1}}{1-z}\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right]$ converges as well.

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## 1.3 Question 6

- Some people are still confused about how to prove continuity.
- For their sake, we repeat the one-line proof.
- *Claim*: If $f:X\to Y$ is Lipschitz-continuous then it is also continuous.
  *Proof*:
  – Recall that $f$ is continuous iff $\forall x_0\in X$ and $\forall\varepsilon>0$ there exists some $\delta(\varepsilon,\,x_0)>0$ such that if $x\in X$ is such that $d_X(x,\,x_0)<\delta(\varepsilon,\,x_0)$ then it follows that $d_Y(f(x),\,f(x_0))<\varepsilon$.
  – Some people are not quite sure how to "parse" this logical condition into a recipe of how to prove continuity. For their sake I repeat here the recipe:
    1. You meet someone on the street.
    2. They write on a piece of paper some $x_0\in X$ and also some $\varepsilon>0$. You don't know in advance what they are going to write. Your recipe better work *no matter which* $x_0$ *and* $\varepsilon$ *they choose!*
    3. You are clever enough to then write back to them on a piece of paper some new magic positive number, $\delta(\varepsilon,\,x_0)$ (the brackets here denote that this new number that you have produced will in general be different every time you meet someone new) which then obeys the condition for continuity:

$$\boxed{\left[d_X(x,\,x_0)<\delta(\varepsilon,\,x_0)\Longrightarrow d_Y(f(x),\,f(x_0))<\varepsilon\right]\forall x\in X}$$

  – After this little detour on logic, we proceed to prove the continuity of $f$, given its Lipschitz-continuity.
  – Because $f$ is Lipschitz continuous, we know that $\exists L\geq 0$ such that $d_Y(f(x),\,f(y))\leq L\,d_X(x,\,y)$ for all $(x,\,y)\in X^2$. Note that some people did not fully internalize that $L$ cannot depend on $(x,\,y)$.
  – So pick $\delta(\varepsilon,\,x_0):=\frac{\varepsilon}{L}$.
  – Why does this work? Let $x\in X$ be given such that $d_X(x,\,x_0)<\delta(\varepsilon,\,x_0)$. That means that $d_X(x,\,x_0)<\frac{\varepsilon}{L}$.
  – But then using Lipschitz-continuity we have that $d_Y(f(x),\,f(x_0))<L\,d_X(x,\,x_0)<L\frac{\varepsilon}{L}=\varepsilon$.

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- *Claim*: $\sqrt{\cdot}:[0,\,1]\to\mathbb{R}$ is not Lipschitz-continuous.
  *Proof*:
  – Assume otherwise, that is, assume that $\exists L\geq 0$ such that $|f(x)-f(y)|\leq L|x-y|$ for all $(x,\,y)\in[0,\,1]^2$.
  – Since this is supposed to work for every $(x,\,y)\in[0,\,1]^2$, pick $y=0$ and some $x_0\in(0,\,1]$.
  – Then we have $|f(x_0)-f(y)|=\left|\sqrt{x_0}\right|=\sqrt{x_0}$ whereas $|x_0-y|=x_0$.
  – Then due to our supposed Lipschitz-continuity, we have that $\sqrt{x_0}\leq Lx_0$ or $L\geq\frac{1}{\sqrt{x_0}}$.

- But $x_0$ we arbitrary, we may take the limit $x_0 \to 0$ to obtain that

$$\limsup_{x_0 \to 0} L \geq \limsup_{x_0 \to 0} \frac{1}{\sqrt{x_0}}$$

- Now, $L$ cannot depend on $x_0$ (or $y$) so that the result of the left hand side limit is just $L$.
- On the other hand, we know that $\limsup_{x_0 \to 0} \frac{1}{\sqrt{x_0}} = \infty$.
- Thus we obtain that $L \geq \infty$. In particular, this implies that $L \notin \mathbb{R}$, which is a contradiction with the assumption of Lipschitz-continuity.

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- *Claim*: $\sqrt{\cdot} : [0, 1] \to \mathbb{R}$ is continuous.
  *Proof*:
  - Homework or check official solutions if you're not sure.

# 2 Exercise Sheet Number 9

## 2.1 Uniform Continuity

- Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces.
- A function $f : X \to Y$ is said to be *uniformly continuous* iff:
  - $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that $(\forall (x, x') \in X^2, (d_X(x, x') < \delta(\varepsilon)$ implies $d_Y(f(x), f(x')) < \varepsilon))$.
- Examples:
  - The function $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ is not uniformly continuous.
    *Proof*:
    * Assume otherwise.
    * Then take some $x \in \mathbb{R}$ and define $x' \in \mathbb{R}$ by $x' := x + \frac{1}{2}\delta(\varepsilon)$.
    * Then

$$\begin{aligned} |f(x) - f(x')| &= \left| f(x) - f\left(x + \frac{1}{2}\delta(\varepsilon)\right)\right| \\ &= \left| x^2 - \left(x^2 + x\delta(\varepsilon) + \frac{1}{4}\delta(\varepsilon)^2\right)\right| \\ &= \delta(\varepsilon)\left|x + \frac{1}{4}\delta(\varepsilon)\right| \\ &\geq \delta(\varepsilon)\left||x| - \frac{1}{4}\delta(\varepsilon)\right| \end{aligned}$$

    * Thus, we see that if we pick $x$ such that $x > \frac{\varepsilon}{\delta(\varepsilon)} + \frac{1}{4}\delta(\varepsilon)$ then obviously $|f(x) - f(x')| > \varepsilon$ and we reach a contradiction. This hinged on the fact that $\delta(\varepsilon)$ cannot depend on $x$.

    ■

  - *Claim*: If $f : X \to Y$ is Lipschitz-continuous then it is uniformly continuous.
    *Proof*:
    * Use the same proof as above (for the solution of question 6 from homework sheet number 7) and note that $\delta(\varepsilon)$ did not depend on the points $x$ or $x'$.

    ■

## 2.2 Compactness

### 2.2.1 An Open Cover

Let $X$ be a space with $Open(X)$ somehow given on it (via a metric or otherwise (in a way we haven't seen). A subset $\mathcal{A}$ of $Open(X)$ (which is in turn a set of subsets of $X$, where each subset is open) is called an open cover of $X$ iff

$$\boxed{X = \bigcup \mathcal{A}}$$

### 2.2.2 Actual Definition

- A space $X$ with $Open\left(X\right)$ defined on it is called *compact* iff every open cover $\mathcal{A}$ admits a finite open subcover.
  - That is, $\forall\mathcal{A}\subseteq Open\left(X\right)$ such that $X=\bigcup\mathcal{A}$, $\exists n\in\mathbb{N}$ and some elements $A_i\in\mathcal{A}$ for all $i\in\{\,1,\,\ldots,\,n\,\}$ such that
$$X=\bigcup_{i=1}^{n}A_i$$
- Examples:
  - $[0,\,1]$ is compact.
  - $\mathbb{R}$ is not compact.

### 2.2.3 The Diameter of a Set

Let a metric space $(X,\,d_X)$ be given, and let $A\subseteq X$. Then the diameter of $A$ is defined as the number $\sup\left(\{\,d_X\left(a,\,a'\right)\mid(a,\,a')\in A\,\}\right)$.

### 2.2.4 Heine-Borel Theorem

- *Claim*: A subset $A\subseteq\mathbb{R}^n$ where $\mathbb{R}^n$ is taken with the Euclidean metric, is compact if and only if $A\in Closed\left(\mathbb{R}^n\right)$ and $diam\left(A\right)<\infty$.
  *Proof*: (in Colloquium of Week 10)

## 2.3 The Lebesgue Number

- Let $\mathcal{A}$ be an open covering of the metric space $(X,\,d_X)$.
- *Claim*: If $X$ is compact, then $\exists$ some $\alpha_{\mathcal{A}}>0$ such that $(\forall S\in2^X$ such that $diam\left(S\right)<\alpha_{\mathcal{A}})$ $\exists$ some $A_S\in\mathcal{A}$ such that $S\subseteq A_S$.
  *Proof*:
  - Case 1: If it happens that the whole space $X$ is in the open covering ($X\in\mathcal{A}$), then any positive number is the Lebesuge number, and then $A_S:=X$ always.
  - Case 2:
    * Because $X$ is compact, $\exists$ a finite open sub cover of $\mathcal{A}$, $(A_i)_{i=1}^n$ for some $n$, where each $A_i\in\mathcal{A}$ for all $i\in\{\,1,\,\ldots,\,n\,\}$.
    * Define $C_i:=X\backslash A_i$ for all $i\in\{\,1,\,\ldots,\,n\,\}$.
    * Define $f:X\to\mathbb{R}$ by
$$x\mapsto\frac{1}{n}\sum_{i=1}^{n}d_X\left(x,\,C_i\right)$$
      where the distance of a point from a set is defined as $d_X\left(x,\,A\right)\equiv\inf\left(\{\,d_X\left(x,\,a\right)\mid a\in A\,\}\right)$.
    * *Claim*: $f\left(X\right)\cap\{0\}=\varnothing$ (that is, $f$ never evaluates to 0)
      *Proof*:
      · Let $x\in X$ be given. Then $\exists i_x\in\{\,1,\,\ldots,\,n\,\}$ such that $x\in A_{i_x}$ (possible as that is a cover).
      · Choose some $\varepsilon>0$ such that $B_{d_X}\left(x,\,\varepsilon\right)\subseteq A_{i_x}$ (possible as $A_{i_x}\in Open\left(X\right)$).
      · Thus $d_X\left(x,\,C_{i_x}\right)\geq\varepsilon$ as $C_{i_x}\equiv X\backslash A_{i_x}$, and so,
$$\begin{aligned}f\left(x\right)&\equiv\frac{1}{n}\sum_{i=1}^{n}d_X\left(x,\,C_i\right)\\&=\frac{1}{n}d_X\left(x,\,C_{i_x}\right)+\frac{1}{n}\sum_{i=1\wedge i\neq i_x}^{n}d_X\left(x,\,C_i\right)\\&\geq\frac{\varepsilon}{n}+\frac{1}{n}\sum_{i=1\wedge i\neq i_x}^{n}d_X\left(x,\,C_i\right)\\&\geq\frac{\varepsilon}{n}\\&>0\end{aligned}$$
    * Because $f$ is continuous on a *compact* set, there exists a minimum value:
$$\boxed{\alpha_{\mathcal{A}}:=\min\left(f\left(X\right)\right)}$$
    * *Claim*: $\alpha_{\mathcal{A}}$ is really the Lebesgue-number of $\mathcal{A}$.
      *Proof*:

- Let $S \in 2^X$ be given such that $diam(S) < \alpha_{\mathcal{A}}$.
- Take some $x_0 \in S$. Then $S \subseteq B_{d_X}(x_0, \alpha_{\mathcal{A}})$ by definition of the diameter of a set.
- Define $m \in \{1, \ldots, n\}$ such that $d_X(x_0, C_m)$ is the largest of all the numbers $d_X(x_0, C_i)$.
- Then $f(x_0) \leq d_X(x_0, C_m)$ by the definition of $f$, yet we have $\alpha_{\mathcal{A}} \leq f(x_0)$ by the definition of $\alpha_{\mathcal{A}}$, so that $\alpha_{\mathcal{A}} \leq d_X(x_0, C_m)$.
- From which we obtain that $B_{d_X}(x_0, \alpha_{\mathcal{A}}) \subseteq X \backslash C_m = A_m \in \mathcal{A}$. But we had $S \subseteq B_{d_X}(x_0, \alpha_{\mathcal{A}})$, so we are done.

## 2.4 The Heine-Cantor Theorem

Let $f : X \to Y$ be a continuous function between the two metric spaces $(X, d_X)$ and $(Y, d_Y)$ and assume further that $X$ is compact. Then $f$ is actually not just continuous, but in fact *uniformly continuous*.
*Proof*:

- Let $\varepsilon > 0$ be given.
- Let $\mathcal{B}$ be the open covering $Y$ by $B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)$.
- Let $\mathcal{A}$ be the open covering of $X$ by $f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right)$. It *is* indeed an open covering because $f$ is continuous, and $B_{d_Y}\left(y, \frac{\varepsilon}{2}\right) \in Open(Y)$ of course.
- Let $\alpha_{\mathcal{A}}$ be the Lebesgue number of $\mathcal{A}$.
- Take any $(x, x') \in X^2$ such that $d_X(x, x') < \alpha_{\mathcal{A}}$.
- Then $diam(\{x, x'\}) < \alpha_{\mathcal{A}}$, so that we can apply the condition of the Lebesgue number on the set $\{x, x'\}$: $\exists$ some $A \in \mathcal{A}$ such that $\{x, x'\} \subseteq A$.
- But all sets $A \in \mathcal{A}$ have the form $f^{-1}\left(B_{d_Y}\left(y, \frac{\varepsilon}{2}\right)\right)$, so that there must exist some $y_A \in Y$ such that $A = f^{-1}\left(B_{d_Y}\left(y_A, \frac{\varepsilon}{2}\right)\right)$.
- As a result we have that $\exists y_A \in Y$ such that $\{x, x'\} \subseteq f^{-1}\left(B_{d_Y}\left(y_A, \frac{\varepsilon}{2}\right)\right)$, which implies that $f(\{x, x'\}) \subseteq B_{d_Y}\left(y_A, \frac{\varepsilon}{2}\right)$.
- This in turn implies that $\begin{cases} d_Y(x, y_A) < \frac{\varepsilon}{2} \\ d_Y(x', y_A) < \frac{\varepsilon}{2} \end{cases}$. Add those two inequalities to obtain $d_Y(x, y_A) + d_Y(x', y_A) < \varepsilon$.
- But of course, $d_Y(x, x') < d_Y(x, y_A) + d_Y(x', y_A)$ by the triangle inequality.
- Thus we finally obtain that $d_Y(x, x') < \varepsilon$.

## 2.5 Concrete Tips for the Exercises

### 2.5.1 Question 2

- All sums are finite, so you may freely write $\cos(kx) = \Re(\exp(ikx))$ and simiarly for $\sin(kx)$.
- Use $\exp(ikx) = [\exp(ix)]^k$.
- Use the geometric sum formula.

### 2.5.2 Question 3

- Write the permiter as $\sum_{k=0}^{n-1} |p_{k+1}(x) - p_k(x)|$.
- Use given tip.
- For part (b) use the 8th recitation session proof of the limit.
- For part (b) then write $L_n(x) = |x| \left|\frac{2n}{x} \sin\left(\frac{x}{2n}\right)\right|$.

### 2.5.3 Question 4

- Part (a) is completely trivial negation of the definition of uniform continuity.
- Part (b): use the property of compact sets (what does that mean about sequences in them?)

### 2.5.4 Question 5

- This is easy, just invert all the axioms of open sets and use the fact that a set is closed if its complement is open.
- For part (b) i. Show that the complement is open, using the definition of complement set in one of the previous recitaionts (that it contains an open-ball).
- For part (b) ii. Write $C = [0, 1] \cap \left(\bigcap_{k \in \mathbb{N}} A_k\right)$ where $A_k = 3^{-k} A_0$ where $A_0 = \bigcup_{n \in \mathbb{Z}} [2n, 2n+1]$. Why does this define the Cantor set? Why can we now use the Heine-Borel theorem?

### 2.5.5 Question 6

- For part (a): Make the definitions of the tip, and then prove that $I_x \subseteq U$ by assuming the converse and lead to a contradiction.
- For part (b): Just follow your nose (there are no special tricks in this question).
- For part (c): Define $\mathcal{I} := \{ I_x \mid x \in U \}$. Then clearly $\bigcup \mathcal{I} = U$. Define $\hat{\mathcal{I}} := \{ I \cap \mathbb{Q} \mid I \in \mathcal{I} \}$ (only the rational part of each set $I \in \mathcal{I}$), and define some $f : \hat{\mathcal{I}} \to \mathbb{Q}$ which obeys the following condition: $f(I \cap \mathbb{Q}) \in I \cap \mathbb{Q}$. How to define such a function? How does it help?