

ANALYSIS 2

RECITATION SESSION OF WEEK 2

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1. LOGISTICS

Fact 1.1. Recitation Sessions are held every Monday 13-15 at HG D 5.2. The goal of these sessions is to prepare you for solving the current week's homework, go over the solutions from the previous week's homework, and perhaps clarify unresolved problems from the lecture.

Fact 1.2. Colloquium Sessions—either new material, review, or left over from recitation sessions—are held every Thursday 14-15 at LFW C 1.

Fact 1.3. The best way to reach me is via my email jshapiro@phys.ethz.ch. You can also find the summaries of the lessons on my website at <http://www.phys.ethz.ch/~jshapiro/>. You may inquire about any matter, including but not limited to: difficulties with the lecture, difficulties with the recitation sessions, difficulties with the homework, and the meaning of life.

Fact 1.4. The most straight forward way to do well in this class is to do the homework every week. Another possibility is to do the homework all at once just before the exam; you might still get a good grade, but you will have shortened your lifespan with heightened anxiety.

If you want to have your homework reviewed, hand it in by the end of the recitation session, on Monday, to me. You will get your work back the following week. Your work will be reviewed, but not graded, and will not affect your credit or final mark.

2. CHRISTMAS EXERCISE SHEET REVIEW

2.1. Differential Equations. Recall that a differential equation is a condition where the unknown is not a number, but rather, a map.

For example, find all maps that obey the condition

$$y'' - 5y' + 6y = 0 \tag{2.1}$$

We can think of a lucky guess, $y(x) := e^{tx}$ where now t is unknown. This approach seems random, but in fact it's the usual way to solve differential equations: guessing.

Plug in our guess, to get a relation on t (instead of on y):

$$t^2 - 5t + 6 = 0$$

so that $t = \frac{5 \pm \sqrt{25-24}}{2} = \begin{cases} 2 \\ 3 \end{cases}$. Thus both $y = e^{2x}$ and $y = e^{3x}$ solve equation (2.1).

Claim 2.1. $x \mapsto e^{2x}$ and $x \mapsto e^{3x}$ are linearly independent.

Proof. We say that they are linearly independent in exactly the same way two ordinary \mathbb{R}^n vectors are independent: the solution to the equation

$$\alpha e^{2x} + \beta e^{3x} = 0 \forall x \in \mathbb{R} \tag{2.2}$$

with two reals α and β is necessarily $\alpha = 0$ and $\beta = 0$. Let us see why. equation (2.2) implies $\alpha + \beta e^x = 0 \forall x \in \mathbb{R}$, in particular, plug in $x = 0$ to get $\alpha = -\beta$ so that we should have $\beta(e^x - 1) = 0 \forall x \in \mathbb{R}$, which is only possible when $\beta = 0$. \square

Furthermore, equation (2.1) has an important property, called *linearity*: the operator $C^2(\mathbb{R}^{\mathbb{R}}) \ni y \mapsto y'' - 5y' + 6y \in \mathbb{R}^{\mathbb{R}}$ is linear (it preserves addition and scalar multiplication) and as such its image is a vector subspace (the vectors being functions). equation (2.1) is a second order differential equation (the highest derivative is a second derivative) and so it should have two linearly independent solutions (for more information about this claim, see [1]). Thus the dimension of this vector space is 2 and as we have found two linearly independent solutions, we can write the most general element in the vector space as a linear combination of these two:

$$y_{\text{generic}}(x) = c_1 e^{2x} + c_2 e^{3x}$$

You should think of the solution space of equation (2.1) like \mathbb{R}^2 , and to describe a general point in it, we need to specify “how much we are going” in each direction (\hat{x} or \hat{y}). So a general vector in \mathbb{R}^2 is written as $c_1 \hat{x} + c_2 \hat{y}$. With differential equations, geometric directions are replaced by “directions in an abstract function space specified by independent solutions: e^{2x} and e^{3x} .”

Another property of equation (2.1) is that it is *ordinary*, in the sense that the function we are dealing with, y , is a function only one variable x . However, there are also partial differential equations, where there are two or more variables to each map (the unknown) and the derivatives can be with respect to any of these variables.

In general, it might be helpful to keep in mind the following taxonomy of differential equations with example of each:

	linear	non-linear
ordinary	$y''(x) - y(x) = 0$	$y'(x) - \sin[y(x)] = 0$
partial	$\partial_x y(x, t) + \partial_t y(x, t) + y(x, t) = 0$	$\partial_x y(x, t) + [\partial_t y(x, t)]^2 = 0$

2.1.1. *Question 4.* Let $n \in \mathbb{N} \setminus \{0\}$ and $(a_0, \dots, a_{n-1}) \in \mathbb{R}^n$ be given. Then define the differential operator $\mathcal{D} : C^n \rightarrow \mathbb{R}$ given by

$$\mathcal{D}(y) := y^{(n)} + \sum_{j=n-1}^0 a_j y^{(j)}$$

Then

$$\mathcal{D}(y) = 0 \tag{2.3}$$

is a linear differential equation of order n . Define

$$\mathcal{L} := \{y \in C^n \mid \mathcal{D}(y) = 0\}$$

as the space of all solutions of equation (2.3).

Claim 2.2. The n solutions $\{y_j\}_{j=1}^n \subset \mathcal{L}$ are a basis of \mathcal{L} iff the matrix defined by the components $y_j^{(i)}(x)$ is in $GL_n(\mathbb{R})$ for some $x \in \mathbb{R}$; Furthermore, if $y_j^{(i)}(x) \in GL_n(\mathbb{R})$ for some $x \in \mathbb{R}$ then $y_j^{(i)}(x) \in GL_n(\mathbb{R})$ for all $x \in \mathbb{R}$.

Proof. $\implies \forall x \in \mathbb{R}$, define the map

$$\mathcal{L} \ni y \mapsto \mathcal{A}_x(y) = (y(x), y'(x), \dots, y^{(n-1)}(x)) \in \mathbb{R}^n$$

This map is clearly linear. Recall Koenigsberger 101. Satz 1:

Claim. $\forall x \in \mathbb{R}$, the map \mathcal{A}_x is injective.

We know that linear transformations which are injective preserve linear independence. As $\{y_j\}_{j=1}^n$ is assumed to be a basis, it is linearly independent. That means that all of the vectors $\mathcal{A}_x(y_j)$ are linearly independent. But the matrix $y_j^{(i)}(x)$ has as its columns the vectors $\mathcal{A}_x(y_j)$. But a matrix with linearly independent columns is invertible. So $y_j^{(i)}(x)$ is invertible.

\implies If for some $x_0 \in \mathbb{R}$, $y_j^{(i)}(x_0)$ is invertible, then its columns are linearly independent. Thus, by the injectivity of \mathcal{A}_{x_0} , it follows that $\{y_j\}_{j=1}^n$ are linearly independent and so they are a basis of \mathcal{L} . To complete the claim, use the opposite direction now to show the invertibility of $y_j^{(i)}(x)$ for any other $x \in \mathbb{R}$. \square

Claim 2.3. If the characteristic polynomial

$$p_n(\lambda) \equiv \lambda^n + \sum_{j=n-1}^0 a_j \lambda^{n-1} \tag{2.4}$$

has n -different roots $\{\lambda_j\}_{j=1}^n$, then $\{x \mapsto e^{\lambda_j x}\}_{j=1}^n$ is a basis for \mathcal{L} .

Proof. First of all, plugging $e^{\lambda x}$ into equation (2.3) gives the characteristic polynomial equation (2.4), so that $\mathcal{D}(x \mapsto e^{\lambda x}) = 0$ indeed and so $(x \mapsto e^{\lambda x}) \in \mathcal{L}$. Secondly, for any two different λ and $\tilde{\lambda}$, $x \mapsto e^{\lambda x}$ and $x \mapsto e^{\tilde{\lambda} x}$ are linearly independent (similarly to how we showed this above in the special case). Thus, a set of n different solutions to the characteristic polynomial gives a set of n different linearly independent solutions and thus a basis of \mathcal{L} . \square

Claim 2.4. Let $q \in \mathbb{R}^{\mathbb{R}}$, $\{y_j\}_{j=1}^n$ be a basis of \mathcal{L} , and let $z(x)$ be the inverse of the matrix defined by the components $y_j^{(i)}(x)$ for every $x \in \mathbb{R}$. (Small note: the index j runs from 1 to n and the index i runs from 0 to $n-1$, just for convenience, but this doesn't affect any of the matrix manipulations). Then

$$x \mapsto \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j(x) \quad (2.5)$$

is a particular solution to the equation

$$\mathcal{D}(y) = q \quad (2.6)$$

Proof. Observe that by definition,

$$\begin{aligned} \sum_{j=1}^n y_j^{(i)}(x) [z_{j,n-1}(x) q(x)] &= \mathbb{1}_{i,n-1} q(x) \\ &= \delta_{i,n-1} q(x) \end{aligned}$$

Define

$$y_p(x) := \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j(x)$$

$$\text{Claim 2.5. } y_p^{(k)}(x) = \begin{cases} \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(k)}(x) & k \in \mathbb{Z}_n \\ \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(n)}(x) + q(x) & k = n \end{cases}$$

Proof. Use induction on k . For $k = 0$ the statement is true by definition of $y_p(x)$. Assume it is true for some $k < n-1$, check $k+1$:

$$\begin{aligned} \left(\sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(k)}(x) \right)' &= \sum_{j=1}^n \left\{ z_{j,n-1}(x) q(x) y_j^{(k)}(x) + \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(k+1)}(x) \right\} \\ &= \underbrace{\sum_{j=1}^n y_j^{(k)}(x) z_{j,n-1}(x) q(x)}_{\delta_{k,n-1}=0} + \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(k+1)}(x) \\ &= \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(k+1)}(x) \end{aligned}$$

To obtain $k = n$ we differentiate once more:

$$\begin{aligned} \left(\sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(n-1)}(x) \right)' &= \sum_{j=1}^n \left\{ z_{j,n-1}(x) q(x) y_j^{(n-1)}(x) + \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(n)}(x) \right\} \\ &= q(x) + \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(n)}(x) \end{aligned}$$

Now apply \mathcal{D} on y_p to obtain:

$$\begin{aligned} \mathcal{D}(y_p) &= y_p^{(n)} + \sum_{j=n-1}^0 a_j y_p^{(j)} \\ &= q(x) + \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] y_j^{(n)}(x) + \\ &\quad + \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] \\ &= q(x) + \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] \left[y_j^{(n)}(x) + \sum_{r=n-1}^0 a_r y_j^{(r)}(x) \right] \\ &= q(x) + \sum_{j=1}^n \left[\int_0^x z_{j,n-1}(x') q(x') dx' \right] \underbrace{\left[\mathcal{D}(y_j)(x) \right]}_0 \\ &= q(x) \end{aligned}$$

2.1.2. Question 5.

Claim 2.6. The general real solution to

$$y^{(4)}(x) - y(x) = 0$$

is given by

$$c_1 e^x + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x)$$

with all coefficients real.

Proof. Make the Ansatz $y(x) = e^{\lambda x}$ to get the characteristic polynomial

$$\lambda^4 - 1 = 0$$

which has the solution set $\{1, -1, i, -i\}$. Thus the four linearly independent solutions are $\{e^x, e^{-x}, e^{ix}, e^{-ix}\}$, and the most general solution is given by

$$y_{\text{general}}(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + c_4 e^{-ix}$$

where in general, $c_j \in \mathbb{C}$. If we want only real solutions then we insist on $c_4 = \bar{c}_3$ and $(c_1, c_2) \in \mathbb{R}^2$ so

$$\begin{aligned} y_{\text{general}}^{\text{real}}(x) &= c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + \bar{c}_3 e^{-ix} \\ &= c_1 e^x + c_2 e^{-x} + c_3 [\cos(x) + i \sin(x)] + \bar{c}_3 [\cos(x) - i \sin(x)] \\ &= c_1 e^x + c_2 e^{-x} + \underbrace{2\Re\{c_3\}}_{\tilde{c}_3} \cos(x) + \underbrace{2\Im\{c_3\}}_{\tilde{c}_4} \sin(x) \\ &= c_1 e^x + c_2 e^{-x} + \tilde{c}_3 \cos(x) + \tilde{c}_4 \sin(x) \end{aligned}$$

with $(c_1, c_2, \tilde{c}_3, \tilde{c}_4) \in \mathbb{R}^4$. □

2.1.3. Question 6.

Claim 2.7. The general real solution for $x > 0$ to

$$x^2 \partial_x^2 y(x) - 3x \partial_x y(x) + 5y(x) = 0 \tag{2.7}$$

is given by

$$y(x) = c_1 x^2 \cos(\log(x)) + c_2 x^2 \sin(\log(x))$$

Proof. Define $h(t) := y(e^t)$ and make the change of variable association $x = e^t$. Then by the chain rule

$$\begin{aligned} \partial_t h(t) &= \partial_t y(e^t) \\ &= [\partial_x y(x)] \partial_t e^t \\ &= [\partial_x y(x)] e^t \end{aligned}$$

so that

$$\partial_x y(x) = e^{-t} \partial_t y(e^t) = e^{-t} \partial_t h(t)$$

and

$$\begin{aligned} \partial_t^2 h(t) &= \partial_t^2 y(e^t) \\ &= \partial_t [e^t \partial_x y(x)] \\ &= e^t \partial_x y(x) + e^t \partial_x^2 y(x) e^t \\ &= \partial_t h(t) + e^{2t} \partial_x^2 y(x) \end{aligned}$$

so that

$$\begin{aligned} \partial_x^2 y(x) &= e^{-2t} [\partial_t^2 h(t) - \partial_t h(t)] \\ &= e^{-2t} \partial_t^2 h(t) - e^{-2t} \partial_t h(t) \end{aligned}$$

We plug in these changes into equation (2.7) to get

$$e^{2t} e^{-2t} [\partial_t^2 h(t) - \partial_t h(t)] - 3e^t e^{-t} \partial_t h(t) + 5h(t) = 0$$

and so the differential equation to solve is

$$\partial_t^2 h(t) - 4\partial_t h(t) + 5h(t) = 0$$

which gives $h(t) = c_1 e^{2t} \cos(t) + c_2 e^{2t} \sin(t)$ and so

$$\begin{aligned} y(x) &\stackrel{x \geq 0}{=} h(\log(x)) \\ &= c_1 x^2 \cos(\log(x)) + c_2 x^2 \sin(\log(x)) \end{aligned}$$

□

2.1.4. Question 7.

Claim 2.8. The general solution to the equation

$$y'(t) = a(t)y(t) - b(t) \quad (2.8)$$

is given by

$$y(t) = e^{\mathcal{A}(t)} \left[B - \int e^{-\mathcal{A}(t)} b(t) dt \right]$$

where B is some constant and $\mathcal{A}(t) \equiv \int a(t) dt$

Proof. Start out by finding the solution to the homogeneous equation:

$$y'(t) = a(t)y(t)$$

$$\frac{dy}{y} = a(t) dt$$

$$\log(|y|) = \int a(t) dt + C$$

$$|y| = e^C e^{\int a(t) dt}$$

$$y_{\text{homo}}(t) = \underbrace{\pm e^C}_A e^{\int a(t) dt}$$

$$y_{\text{homo}}(t) = A e^{\int a(t) dt}$$

$$y_{\text{homo}}(t) = A e^{\mathcal{A}(t)}$$

To find a particular solution, we use the method outlined above for the general case (now we have $n = 1$) and so

$$y_{\text{particular}}(t) = \left\{ \int \frac{-b(t)}{A e^{\mathcal{A}(t)}} dt + \tilde{B} \right\} A e^{\mathcal{A}(t)}$$

One may verify that this is indeed a particular solution:

$$\begin{aligned} y'_{\text{particular}}(t) &= \frac{-b(t)}{A e^{\mathcal{A}(t)}} A e^{\mathcal{A}(t)} + \left\{ \int \frac{-b(t)}{A e^{\mathcal{A}(t)}} dt + \tilde{B} \right\} A e^{\mathcal{A}(t)} \underbrace{\mathcal{A}'(t)}_{a(t)} \\ &= -b(t) + y_{\text{particular}}(t) a(t) \end{aligned}$$

which indeed fulfills equation (2.8). Thus, the most general solution is

$$\begin{aligned} y_{\text{general}}(t) &= y_{\text{homo}}(t) + y_{\text{particular}}(t) \\ &= A e^{\mathcal{A}(t)} + \left\{ \int \frac{-b(t)}{A e^{\mathcal{A}(t)}} dt + \tilde{B} \right\} A e^{\mathcal{A}(t)} \\ &= e^{\mathcal{A}(t)} \left\{ \underbrace{A(1 + \tilde{B})}_B - \int e^{-\mathcal{A}(t)} b(t) dt \right\} \\ &= e^{\mathcal{A}(t)} \left[B - \int e^{-\mathcal{A}(t)} b(t) dt \right] \end{aligned}$$

□

3. EXERCISE SHEET NUMBER 1 PREPARATION

3.1. Question 1.

- Recall the notion of equivalence relation (three conditions must hold).

3.2. Question 4.

REFERENCES

- [1] V. I. Arnold. *Ordinary differential equations*. MIT Press, Cambridge, 1973.