ANALYSIS 2
RECITATION SESSION OF WEEK 10

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1. Interior, Closure and Boundary

Let X be a general topological space, and $A \subseteq X$.

1.1. **Example.** $\partial\partial A \neq \partial A$.

Proof. Recall the example where $A = \mathbb{Q}$ and $X = \mathbb{R}$. Then $\partial A = \mathbb{R}$, and so $\partial\partial A = \partial\mathbb{R} = \varnothing$. So we have $\varnothing \neq \mathbb{R}$. □

1.2. **Example.** $\partial\left(\overline{A}\right) \neq \partial A$.

Proof. Again with the example where $A = \mathbb{Q}$ and $X = \mathbb{R}$, we have $\overline{A} = \mathbb{R}$ and so $\partial\left(\overline{A}\right) = \varnothing$ yet $\partial A = \mathbb{R}$! □

1.3. **Example.** $A = \left\{ x \in \mathbb{R}^2 \mid (x)_2 = 0 \right\}$ and $X = \mathbb{R}^2$. Note that $A \in \mathtt{Closed}(X)$ (see this by drawing open balls in the complement). Thus $\overline{A} = A$. Also note that $A^\circ = \varnothing$ (see this by drawing open balls). As a result, $\partial A = A$.

1.4. **Example.** $A = \left\{ x \in \mathbb{R}^2 \mid (x)_2 > 0 \right\}$ and $X = \mathbb{R}^2$. Then $A \in \mathtt{Open}(X)$ (draw open balls) so that $A^\circ = A$. $\overline{A} = \left\{ x \in \mathbb{R}^2 \mid (x)_2 \geqslant 0 \right\}$ ($\overline{A} \supseteq A$ and every point on the line $(x)_2 = 0$ also belongs to the closure because every open ball around any point in it intersects A). Thus $\partial A = \left\{ x \in \mathbb{R}^2 \mid (x)_2 = 0 \right\}$.

1.5. **Example.** $(A, B) \in [\mathtt{Open}(X)]^2$ such that $A \cap B = \varnothing$. Then $\overline{A} \cap \overline{B} \neq \varnothing$.

Proof. Take $X = \mathbb{R}$ and $A = \left(0, \frac{1}{2}\right)$ and $B = \left(\frac{1}{2}, 1\right)$. Then $A \cap B = \varnothing$ yet $\overline{A} \cap \overline{B} = \left\{ \frac{1}{2} \right\}$. □

1.6. **Example.** $\overline{(A^\circ)} \neq A$.

Proof. Take $A = (0, 1) \cup \{ 2 \}$ and $X = \mathbb{R}$. Then $A^\circ = (0, 1)$ (to see this, try to find an open interval around 2 which is contained in A), and so $\overline{(A^\circ)} = [0, 1]$. □

2. Integrals

2.1. **Multi-Dimensional Integrals.** We follow [1] Chapter 10. This allows a somewhat shorter and more compact presentation of a multi-dimensional integral than with the Jordan measure, which is anyway obsoleted by the Lebesgue measure.

- Let I^k be the closed k-cell in $\mathbb{R}^k$. That means $I^k = \prod_{j \in J_k} \left[a_j, b_j\right]$ where $(a, b) \in \left[\mathbb{R}^k\right]^2$ such that $a_j \leqslant b_j$ for all $j \in J_k$.
- For every $j \in J_k$, define I^j to be the j-cell in $\mathbb{R}^j$ defined by $\prod_{l \in J_j} [a_l, b_l]$.
- Let $f \in C^0\left(I^k, \mathbb{R}\right)$.
- Define $f_k := f$ and $f_{k-1} : I^{k-1} \to \mathbb{R}$ by

$$f_{k-1}(x) := \int_{a_k}^{b_k} f_k(x, y)\, dy \quad \forall x \in I^{k-1}$$

  where the integral is the orindary one-dimensional Riemann integral encountered in the last semester.

2.1. *Claim.* f_{k-1} is continuous on I^{k-1}.

Proof. Observe that f_k is *uniformly* continuous on I^k because I^k is compact (being closed and bounded). Let $x \in I^{k-1}$ be given, and let $\varepsilon > 0$ be given. By uniform continuity, $\exists \delta > 0$ such that if $z \in I^{k-1}$ is such that $\|(x, y) - (z, y)\| < \delta$ then $|f_k(x, y) - f_k(z, y)| < \frac{\varepsilon}{b_k - a_k}$.

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Then for such $z \in I^{k-1}$ we have

$$\begin{aligned} |f_{k-1}(x) - f_{k-1}(z)| &= \left| \int_{a_k}^{b_k} f_k(x, y)\, dy - \int_{a_k}^{b_k} f_k(z, y)\, dy \right| \\ &= \left| \int_{a_k}^{b_k} [f_k(x, y) - f_k(z, y)]\, dy \right| \\ &\leqslant \int_{a_k}^{b_k} |f_k(x, y) - f_k(z, y)|\, dy \\ &\leqslant \frac{\varepsilon}{b_k - a_k} \int_{a_k}^{b_k} dy \\ &= \varepsilon \end{aligned}$$

but

$$\begin{aligned} \|(x, y) - (z, y)\| &= \sqrt{\sum_{j \in J_{k-1}} (x_j - z_j)^2} \\ &= \|x - z\| \end{aligned}$$

□

- As a result, we may repeat this process again and again, to obtain functions $f_j \in C^0(I^j, \mathbb{R})$ for all $j \in J_k$ and such that f_{j-1} is the integral of f_j with respect to x_j over $[a_j, b_j]$.
- After k steps we arrive at a number f_0 which we *define* as the integral of f over I^k:

$$\int_{I^k} f(x)\, dx := \int_{a_k}^{b_k} \left(\int_{a_{k-1}}^{b_{k-1}} \left(\dots \left(\int_{a_1}^{b_1} f(x)\, dx_1 \right) \dots \right) dx_{k-1} \right) dx_k \tag{1}$$

2.2. *Claim.* The left hand side of (1) is independent of the order in which the integrations are made. (Theorem 10.2).

2.3. **Definition.** The support of a function $f : \mathbb{R}^k \to \mathbb{R}$ is

$$\begin{aligned} \operatorname{supp}(f) &:= \overline{f^{-1}(\mathbb{R} \backslash \{0\})} \\ &= \overline{\{ x \in \mathbb{R}^k \mid f(x) \neq 0 \}} \end{aligned}$$

2.4. **Example.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1$. Then $\operatorname{supp}(f) = \overline{\mathbb{R}} = \mathbb{R}$.

2.5. **Example.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $\chi_{\mathbb{Q}}(x) \equiv \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Then $\operatorname{supp}(f) = \overline{\mathbb{Q}} = \mathbb{R}$.

2.6. **Example.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $\chi_{B_1(0)}(x) \equiv \begin{cases} 1 & \|x\| < 1 \\ 0 & \|x\| \geqslant 1 \end{cases}$. Then $\operatorname{supp}(f) = \overline{B_1(0)} = \{ x \in \mathbb{R}^2 \mid \|x\| \leqslant 1 \}$.

2.7. *Remark.* Observe that for the support of a function to be compact, all that is necessary is that it is bounded, due to the fact that it is always closed by definition.

2.8. **Definition.** If $f \in C^1(\mathbb{R}^k, \mathbb{R})$ is such that $\operatorname{supp}(f)$ is compact, then

$$\int_{\mathbb{R}^k} f := \int_{I^k} f(x)\, dx \tag{2}$$

where I^k is any k-cell such that $I^k \supseteq \operatorname{supp}(f)$.

2.9. *Remark.* The defintion in (2) is well defined, that is, it is independent of I^k. This is due to the fact that if $I^k \supseteq \operatorname{supp}(f)$, then of course outside of $\operatorname{supp}(f)$, $f = 0$ and so it does not matter which I^k is picked.

2.10. **Example.** Going back to example 2.6, we have $\operatorname{supp}(f)$ compact, and so for example, $I^2 := [-1, 1]^2 \supseteq \overline{B_1(0)}$. Thus we have

$$\begin{aligned} \int_{\mathbb{R}^2} f &= \int_{[-1,1]^2} f(x)\, dx \\ &= \int_{-1}^{1} \int_{-1}^{1} \chi_{B_1(0)}(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_{-1}^{1} \int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}} dx_1\, dx_2 \\ &= \int_{-1}^{1} 2\sqrt{1 - x_2^2} dx_2 \\ &= \pi \end{aligned}$$

- For all $i \in \mathbb{N}$, assume that $\varphi_i \in C^0(\mathbb{R}, \mathbb{R})$ such that $\operatorname{supp}(\varphi_i) \subseteq \left(2^{-i}, 2^{-(i-1)}\right)$ and $\int_{\mathbb{R}} \varphi_i = 1$.
  - Then $\operatorname{supp}(\varphi_1) \subseteq \left(\frac{1}{2}, 1\right)$, $\operatorname{supp}(\varphi_2) \subseteq \left(\frac{1}{4}, \frac{1}{2}\right)$, $\operatorname{supp}(\varphi_3) \subseteq \left(\frac{1}{8}, \frac{1}{4}\right)$ and so on.
  - Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := \sum_{i \in \mathbb{N}} [\varphi_i(x) - \varphi_{i+1}(x)]\, \varphi_i(y)$.

2.11. *Claim.* $\mathrm{supp}(f)$ is compact in $\mathbb{R}^2$, f is continuous except at $(0,0)$, and $\int dy \int f(x,y)\,dx = 0$ yet $\int dx \int f(x,y)\,dy = 1$. Note that f is unbounded in every neighborhoud of $(0,0)$.

Proof. We first try

$$\begin{aligned}\int f(x,y)\,dx &= \int \sum_{i\in\mathbb{N}} [\varphi_i(x) - \varphi_{i+1}(x)]\,\varphi_i(y)\,dx \\ &= \sum_{i\in\mathbb{N}} [1-1]\,\varphi_i(y) \\ &= 0\end{aligned}$$

Observe that this integration is valid because for each fixed y, $\sum_{i\in\mathbb{N}} [\varphi_i(x) - \varphi_{i+1}(x)]\,\varphi_i(y)$ is a finite sum: $\varphi_i(y) = 0$ if $2^{-i} > y$ or if $i > -\log_2(y)$ (where $y > 0$). On the other side,

$$\begin{aligned}\int f(x,y)\,dy &= \int \sum_{i\in\mathbb{N}} [\varphi_i(x) - \varphi_{i+1}(x)]\,\varphi_i(y)\,dy \\ &= \sum_{i\in\mathbb{N}} [\varphi_i(x) - \varphi_{i+1}(x)] \\ &= \varphi_1(x)\end{aligned}$$

amd again the sum is finite for fixed x for the same reason. Because $\int_{\mathbb{R}} \varphi_i(x)\,dx = 1$ for each $i \in \mathbb{N}$ yet the length of $\mathrm{supp}(\varphi_i)$ is 2^{-i} so that φ_i must get bigger and bigger to maintain the integral condition. As a result, f cannot be bounded near the origin. □

2.2. **Fubini's Theorem.** According to Fubini's theorem,

$$\begin{aligned}\int_{X\times Y} f(x,y)\,d(x,y) &= \int_X \left(\int_Y f(x,y)\,dy\right) dx \\ &= \int_Y \left(\int_X f(x,y)\,dx\right) dy\end{aligned}$$

if $f|_y$ is Riemann integrable as a function of x alone and $f|_x$ as a function of y alone, and f is Riemann integrable.

Using this theorem we may reduce many double and triple integrals to eventually ordinary one dimensional integrals.

2.12. **Exercise.** Define $C = \left\{ x \in \mathbb{R}^3 \;\middle|\; (x_1)^2 + (x_2)^2 \leqslant 1 \wedge x_3 \in [0,1] \right\}$. We are interested in the volume of C, which we claim is given by π.

Proof. We start by computing

$$\begin{aligned}\mathrm{vol}(C) &= \int_C 1\,dx\,dy\,dz \\ &= \int_0^1 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx\,dz\end{aligned}$$

Now we may use Fubini's theorem to write

$$\begin{aligned}\mathrm{vol}(C) &= \int_0^1 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx\,dz \\ &= \int_0^1 \left(\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx\right) dz \\ &= \left(\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx\right)\left(z|_0^1\right) \\ &= \int_{-1}^1 \left(y|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\right) dx \\ &= \int_{-1}^1 \left(2\sqrt{1-x^2}\right) dx\end{aligned}$$

and now we have an ordinary one dimensional integral (equal to π). □

2.13. **Exercise.** Evaluate $\int_0^3 \int_0^{x^3} x^2 y\,dy\,dx$.

Proof. We proceed by

$$
\begin{aligned}
\int_0^3 \int_0^{x^3} x^2 y \, dy \, dx &= \int_{x=0}^{x=3} \int_{y=0}^{y=x^3} x^2 y \, dy \, dx \\
&= \int_{x=0}^{x=3} \left(\int_{y=0}^{y=x^3} x^2 y \, dy \right) dx \\
&= \int_{x=0}^{x=3} \left(x^2 \int_{y=0}^{y=x^3} y \, dy \right) dx \\
&= \int_{x=0}^{x=3} \left(x^2 \frac{1}{2} y^2 \Big|_0^{x^3} \right) dx \\
&= \int_{x=0}^{x=3} \left(x^2 \frac{1}{2} y^2 \Big|_0^{x^3} \right) dx \\
&= \frac{1}{2} \int_{x=0}^{x=3} x^8 \, dx \\
&= \frac{1}{2} \frac{1}{9} x^9 \Big|_0^3 \\
&= \frac{1}{18} 3^9 \\
&= \frac{2187}{2}
\end{aligned}
$$

□

2.14. **Exercise.** Evaluate $\int_{[0,\,\pi]^3} \exp(x+y+z) \, dx dy dz$.

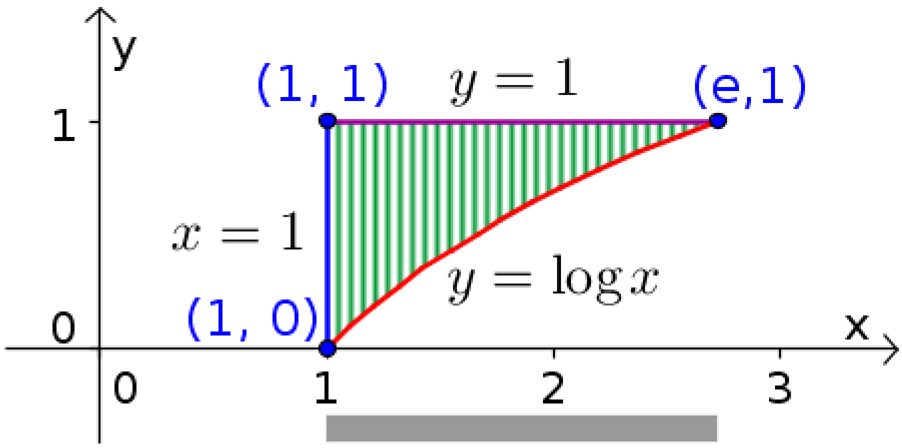
Proof. We start by

$$
\begin{aligned}
\int_{[0,\,\pi]^3} \exp(x+y+z) \, dx dy dz &= \int_0^\pi \int_0^\pi \int_0^\pi \exp(x+y+z) \, dx dy dz \\
&= \int_0^\pi \int_0^\pi \exp(y+z) (e^\pi - 1) \, dy dz \\
&= \int_0^\pi \exp(z) (e^\pi - 1)^2 \, dz \\
&= (e^\pi - 1)^3
\end{aligned}
$$

□

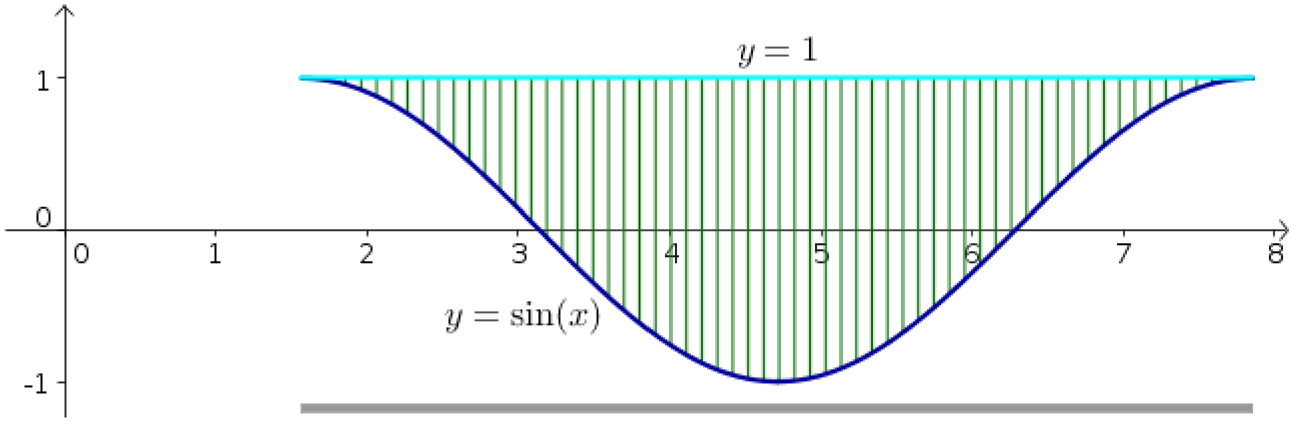
2.3. **Changing the Limits of Integration.**

2.15. **Example.** Change the order of $\int_{y=0}^{1} \int_{x=1}^{e^y} f(x, y) \, dx dy$ to $\int_{x=1}^{e} \int_{y=\log(x)}^{1} f(x, y) \, dy dx$.



Proof.
As the max value of y is 1, we have to integrate x from 1 to $e^y = e^1 = e$. But now y goes from $\log(x)$ to 1. □

2.16. **Example.** Reverse the order of integration from $\int_{\pi/2}^{5\pi/2} \int_{\sin(x)}^{1} f(x, y) \, dy dx$ to $\int_{-1}^{1} \int_{\pi-\arcsin(y)}^{\arcsin(y)+2\pi} f(x, y) \, dx dy$.



Proof.
Now we must be careful about the lower line, because writing simply $x = \arcsin(y)$ will not work as $\arcsin(y)$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and is always increasing. Thus we must separate the lower curve $y = \sin(x)$ into the two curves $x = \pi - \arcsin(y)$ (on the left) and $x = \arcsin(y) + 2\pi$. □

3. Homework Number 8

3.1. Question 1.

- Let $U \in \mathrm{Open}(\mathbb{R}^n)$
- Let $f : U \to \mathbb{R}^n$ be a continuously differentiable vector field on U.
- Let $I(x) \subseteq \mathbb{R}$ be the maximal interval at $x \in \mathbb{R}^n$ for which a solution for the differential equation equation

$$\begin{cases} \gamma_x'(t) & = f(\gamma_x(t)) \\ \gamma_x(0) & = x \end{cases} \quad \gamma_x \in C^1(\mathbb{R}, \mathbb{R}^n) \tag{3}$$

  exists uniquely.
- Define $\Omega := \{ (t, x) \in \mathbb{R} \times U \mid t \in I(x_0) \}$.
- Define $\phi : \Omega \to \mathbb{R}^n$ as the flow of the vector field, that means,

$$\phi(t, x) \quad := \quad \gamma_x(t)$$

  where γ_x is the solution to (3), for all $(t, x) \in I(x) \times U$. That is, we know that

$$\begin{cases} (\partial_t \phi)(t, x) & = f(\phi(t, x)) \\ \phi(0, x) & = x \end{cases} \quad \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$$

- Assume ϕ is continuously differentiable.
- Let $\xi_0 \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ be given.
- Define $\xi : I(x_0) \to \mathbb{R}^n$ by

$$\begin{aligned} \xi(t) : &= ((\partial_x \phi)(t, x_0))(\xi_0) && (4) \\ &= \sum_{i \in J_n} (((\partial_x \phi)(t, x_0))(\xi_0))_i \hat{e}_i && (5) \\ &= \sum_{i \in J_n} \sum_{j \in J_n} (((\partial_x \phi)(t, x_0)))_{ij} (\xi_0)_j \hat{e}_i && (6) \\ &= \sum_{(i,j) \in J_n^2} \left(\left(\partial_{x_j} \phi_i\right)(t, x_0)\right)(\xi_0)_j \hat{e}_i && (7) \end{aligned}$$

3.1. *Claim.* ξ fulfills the differential equation equation $\begin{cases} \xi'(t) & = f'(\phi(t, x_0)) \\ \xi(0) & = \xi_0 \end{cases}$.

Proof. Plug in 0 into (4) to obtain

$$\xi(0) \quad = \quad ((\partial_x \phi)(0, x_0))(\xi_0)$$

but observe that

$$\begin{aligned}
\left(\left(\partial_x \phi\right)(0, x_0)\right) &= \sum_{(i,j)\in J_n^2} \left(\left(\partial_{x_i}\phi_j\right)(0, x_0)\right)\hat{E_{ji}} \\
&\equiv \sum_{(i,j)\in J_n^2} \hat{E_{ji}} \lim_{t\to 0}\frac{\phi_j\left(0, x_0 + t\hat{e}_i\right) - \phi_j\left(0, x_0\right)}{t} \\
&= \sum_{(i,j)\in J_n^2} \hat{E_{ji}} \lim_{t\to 0}\frac{\left(x_0 + t\hat{e}_i\right)_j - \left(x_0\right)_j}{t} \\
&= \sum_{(i,j)\in J_n^2} \hat{E_{ji}}\delta_{ij} \\
&= \sum_{i\in J_n} \hat{E}_{ii} \\
&= \mathbb{1}
\end{aligned}$$

where $\hat{E_{ji}}$ is the unit vector of the matrix with 1 on the jth row and ith column, and zero otherwise.

- Thus, indeed $\xi(0) = \xi_0$.
- Next,

$$\begin{aligned}
\xi'(t) &\equiv \sum_{i\in J_n} \hat{e}_i \left[\left(\partial_t \xi_i\right)(t)\right] \\
&= \sum_{i\in J_n} \hat{e}_i \left[\left(\partial_t \sum_{j\in J_n}\left(\left(\partial_{x_j}\phi_i\right)(t, x_0)\right)\left(\xi_0\right)_j\right)\right] \\
&= \sum_{(i,j)\in J_n^2} \hat{e}_i \left(\partial_t\partial_{x_j}\phi_i(t, x_0)\right)\left(\xi_0\right)_j \\
&\stackrel{*}{=} \sum_{(i,j)\in J_n^2} \hat{e}_i \left(\partial_{x_j}\partial_t\phi_i(t, x_0)\right)\left(\xi_0\right)_j \\
&= \sum_{(i,j)\in J_n^2} \hat{e}_i \left(\left(\partial_{x_j} f_i\circ\phi\right)(t, x_0)\right)\left(\xi_0\right)_j \\
&= \sum_{(i,j)\in J_n^2} \hat{e}_i \left(\sum_{l\in J_n}\left(\left(\partial_{x_l} f_i\right)\circ\phi\right)\left(\partial_{x_j}\phi_l\right)\right)(t, x_0)\left(\xi_0\right)_j \\
&= \sum_{(i,j,l)\in J_n^3} \hat{e}_i \underbrace{\left(\left(\partial_{x_l} f_i\right)\circ\phi\right)(t, x_0)}_{\left(f'(\phi(t,x_0))\right)_{il}} \underbrace{\left(\partial_{x_j}\phi_l\right)(t, x_0)\left(\xi_0\right)_j}_{\xi_l(t)}
\end{aligned}$$

  where in $*$ we have used theorem 9.40 in [1] which states that if $\partial_t\phi$, $\partial_{x_j}\phi$ and $\partial_{x_j}\partial_t\phi$ exist on all point of Ω and $\partial_{x_j}\partial_t\phi$ is continuous at some $(t_0, x_0)\in\Omega$. Then there exists $\left(\partial_t\partial_{x_j}\phi\right)(t_0, x_0)$ which is equal to:

$$\left(\partial_t\partial_{x_j}\phi\right)(t_0, x_0) = \left(\partial_{x_j}\partial_t\phi\right)(t_0, x_0)$$

- Now, As ϕ is assumed to be continuously differentiable, $\partial_t\phi$ and $\partial_{x_j}\phi$ exist. By definition, $\left(\partial_t\phi\right)(t, x) \equiv f(\phi(t, x))$ so that

$$\begin{aligned}
\left(\partial_{x_j}\partial_t\phi\right)(t, x) &= \partial_{x_j} f(\phi(t, x)) \\
&= \sum_{l\in J_n}\left(\partial_{x_l} f\right)(\phi(t, x))\left(\partial_{x_j}\phi_l(t, x)\right)
\end{aligned}$$

  because ϕ is continuously differentiable, f is continuously differentiable, then $\left(\partial_{x_j}\partial_t\phi\right)(t, x)$ exists and is continuous.

□

3.2. **Question 3.**

- Observe it is not necessary to write down what the solution for x would be. Don't make life harder than what it has to be.
- Need to prove $[\exp(A)]^T = \exp\left(A^T\right)$, and $\left[A, A^T\right] = 0$. Both are easy.

3.3. **Question 5.**

- Observe that if $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ then the eigenvalues are $1 \pm i$ and the eigenvectors are $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ so that

$$\begin{aligned}
\exp(At) &= \exp\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} t\right) \\
&= \exp\left(\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} t \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}^{-1}\right) \\
&= \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} \exp\left(\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} t\right)\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \exp((1+i)t) & 0 \\ 0 & \exp((1-i)t) \end{bmatrix} \underbrace{\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}^{-1}}_{-\frac{1}{2}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}} \\
&= -\frac{1}{2}e^t \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{bmatrix}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \\
&= -\frac{1}{2}e^t \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}\begin{bmatrix} i\exp(it) & -\exp(it) \\ i\exp(-it) & \exp(-it) \end{bmatrix} \\
&= -\frac{1}{2}e^t \begin{bmatrix} -\exp(it) - \exp(-it) & -i\exp(it) + i\exp(-it) \\ i\exp(it) - i\exp(-it) & -\exp(it) - \exp(-it) \end{bmatrix} \\
&= e^t \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}
\end{aligned}$$

This is a rotation by t radians counter-clockwise and a dilation by e^t.

References

[1] Walter Rudin. *Principles of Mathematical Analysis (International Series in Pure and Applied Mathematics)*. McGraw-Hill Science/Engineering/Math, 1976.