ANALYSIS 2
HINTS FOR HOMEWORK NUMBER 12

1. Question 1

For this question I recommend reading [1] (A simple search on the web should lead you to a freely available digital version of this book from the various websites that make such files available–email me in doubt). In particular, on page 210 you will find *Lemma 25.1* which states: "If the support of f can be covered a single coordinate patch, the integral $\int_M f$ is well-defined, independent of the choice coordinate patch."

Note that [1] is a very nice textbook for you in general, because it is a very clear presentation of manifolds in $\mathbb{R}^n$ (that is, exactly the submnaifolds you have been talking abuot in this course).

2. Question 2

See *Theorem 21.3* in [1].

3. Question 3

Work with $\rho : \mathbb{R} \to \mathbb{R}$ defined by $\rho(t) := \begin{cases} \exp\left(-\frac{1}{1-t}\right) & t < 1 \\ 0 & t \geqslant 1 \end{cases}$ and use the chain rule. For ρ, note that for $t > 1$, ρ is a constant, and for $t < 1$ use induction to prove a guess for $\rho^{(n)}(t)$.

4. Question 4

4.1. **Part (a).** Use the coordinate chart $\psi : U \to S^2_+$ where $U \equiv \{ (u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1 \}$ given by

$$\psi(u, v) := \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$

Then use the formula

$$\int_{S^2_+} (x + y + z)\, dS = \int_U \left(u + v + \sqrt{1 - u^2 - v^2}\right) \sqrt{\det\left(d\psi(u, v)^T d\psi(u, v)\right)} du dv$$

Your result must be π.

4.2. **Part (b).** Use the same U and $\psi(u, v) := (u, v, u^2 + v^2)$.

Your result must be $\frac{5\sqrt{5}}{12}\pi$.

5. Question 5

5.1. **Part (a).** Use the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) := \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2$. Note that $T_{R,\, a} = f^{-1}(\{ a^2 \})$. Show that a^2 is a regular value of f. Use the regular value theorem.

5.2. **Part (b).** Use the chart $\psi : (0, 2\pi) \times (0, 2\pi) \to T_{R,\, a}$ given by

$$\psi(\alpha, \beta) := (\cos(\alpha)(R + a\cos(b)), \sin(\alpha)(R + a\cos(b)), a\sin(\beta))$$

and the formula

$$vol_2(T_{R,\, a}) \equiv \int_{T_{R,\, a}} 1\, dS = \int_{(0,\, 2\pi)^2} \sqrt{\det\left(d\psi(u, v)^T d\psi(u, v)\right)} d\alpha d\beta$$

References

[1] James R. Munkres. *Analysis On Manifolds (Advanced Books Classics)*. Westview Press, 1997.