ANALYSIS 2
RECITATION SESSION OF WEEK 7

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1. The Inverse Function Theorem and the Implicit Function Theorem

Following [1]:
Recall the following definitions:

1.1. **Definition.** If E and F are two Banach spaces, then

$$\mathcal{L}(E; F) \equiv \left\{ \varphi \in F^E \,\middle|\, \varphi \text{ is linear and } \varphi \text{ is continuous} \right\}$$

1.2. **Definition.** If E and F are two Banach spaces, then

$$\mathrm{Isom}(E; F) \equiv \left\{ \varphi \in \mathcal{L}(E; F) \,\middle|\, \exists \varphi^{-1} \in \mathcal{L}(F; E) \right\}$$

1.3. **Definition.** If E and F are two Banach spaces, $V \in \mathrm{Open}(E)$ and $W \in \mathrm{Open}(F)$ then $f : V \to W$ is a C^k-diffeomorphism iff:

- f is bijective.
- $f \in C^k(V; F)$.
- $f^{-1} \in C^k(W; E)$.

1.4. **Example.** $x \mapsto x^3$ is a homeomorphism $\mathbb{R} \to \mathbb{R}$ but not a diffeomorphism because $x \mapsto x^{\frac{1}{3}}$ is not C^1 (at the origin).

1.5. *Claim.* (Inverse function theorem) Let E and F be two Banach spaces. Let $U \in \mathrm{Open}(E)$. Let $f \in C^1(U; F)$ and let $a \in U$ be such that

$$f'(a) \in \mathrm{Isom}(E; F)$$

then $\exists V \in \mathrm{Open}(E)$ such that $a \in V \subseteq U$ and $\exists W \in \mathrm{Open}(F)$ such that $f(a) \in W$ such that $f \in C^1(V; W)$ is a surjective C^1-diffeomorphism.

1.6. *Remark.* When E and F are finite dimensional, then because they are isomorphic, they must be of the same dimension.

1.7. *Claim.* Let $\{E_i\}_{i=1}^n$ and F be Banach spaces (recall that $\|(e_1, \dots, e_n)\| \equiv \sum_{i=1}^n \|e_i\|$). Let $U \in \mathrm{Open}(E_1 \times \dots \times E_n)$. Let $\varphi \in C^1(U; G)$. Then the partial derivatives of φ are given by $\partial_i \varphi = \varphi' \circ u_i$ where $u_i : E_i \to E_1 \times \dots \times E_n$ is given by $e_i \mapsto (0, 0, \dots, e_i, 0, \dots, 0)$. Observe that $\varphi' \circ u_i : U \to \mathcal{L}(E_i; F)$ because $\varphi'((e_1, \dots, e_n)) \circ u_i$ acts on E_i.

1.8. *Claim.* (Implicit function theorem) Let E, F and G be Banach spaces. Let $U \in \mathrm{Open}(E \times F)$. Let $\varphi \in C^1(U; G)$. Let $(e_0, f_0) \in U$ be given such that $\varphi(e_0, f_0) = 0$. Assume that the partial derivative is an isomorphism: $(\partial_F \varphi)(e_0, f_0) \in \mathrm{Isom}(F; G)$. Then $\exists V \in \mathrm{Open}(E \times F)$ such that $(e_0, f_0) \in V \subseteq U$, $\exists W \in \mathrm{Open}(E)$ such that $e_0 \in W$ and $\exists \psi \in C^1(W; F)$ such that

$$[(e, f) \in V \wedge \varphi(e, f) = 0] \quad \boxed{\Leftrightarrow} \quad [e \in W \wedge f = \psi(e)]$$

and

$$\psi'(e_0) = -[(\partial_F \varphi)(e_0, f_0)]^{-1} \circ (\partial_E \varphi)(e_0, f_0)$$

1.9. *Remark.* Observe again that when F and G are finite dimensional, then because they are isomorphic, they are of the same dimension.

1.10. **Example.** ([2] 9.29) Let $E = \mathbb{R}^3$ and $F = \mathbb{R}^2$ so that $G \approx F = \mathbb{R}^2$. Define $f : \underbrace{\mathbb{R}^3 \times \mathbb{R}^2}_{\approx \mathbb{R}^5} \to \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2e^{x_1} + x_2 y_1 - 4y_2 + 3 \\ x_2 \cos(x_1) - 6x_1 + 2y_1 - y_3 \end{bmatrix}$$

Then observe that

$$\begin{aligned} f\left(\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 2e^0 + 1 \cdot 3 - 4 \cdot 2 + 3 \\ 1 \cos(0) - 6 \cdot 0 + 2 \cdot 3 - 7 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

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Compute $f'\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ (from which we learn that $f\in C^1(\mathbb{R}^5;\mathbb{R}^2)$):

$$\begin{aligned}f'\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right) &= \begin{bmatrix}\partial_{y_1}f_1 & \partial_{y_2}f_1 & \partial_{y_3}f_1 & \partial_{x_1}f_1 & \partial_{x_2}f_1\\ \partial_{y_1}f_2 & \partial_{y_2}f_2 & \partial_{y_3}f_2 & \partial_{x_1}f_2 & \partial_{x_2}f_2\end{bmatrix}\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\\ &= \begin{bmatrix}x_2 & -4 & 0 & 2e^{x_1} & y_1\\ 2 & 0 & -1 & -x_2\sin(x_1)-6f_1 & \cos(x_1)\end{bmatrix}\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\\ &= \begin{bmatrix}1 & -4 & 0 & 2 & 3\\ 2 & 0 & -1 & -6 & 1\end{bmatrix}\end{aligned}$$

so that

$$\begin{aligned}\left(\partial_F f\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\begin{bmatrix}x_1\\x_2\end{bmatrix} &\equiv \left(f'\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\\ &\equiv \left(f'\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\begin{bmatrix}0\\0\\0\\x_1\\x_2\end{bmatrix}\\ &= \begin{bmatrix}2 & 3\\ -6 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\end{aligned}$$

or just $\partial_F f\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2 & 3\\ -6 & 1\end{bmatrix}$ and

$$\begin{aligned}\left(\partial_E f\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} &\equiv \left(f'\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right)\\ &\equiv \left(f'\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right)\begin{bmatrix}y_1\\y_2\\y_3\\0\\0\end{bmatrix}\\ &= \begin{bmatrix}1 & -4 & 0\\ 2 & 0 & -1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\end{aligned}$$

or simply $\partial_E f\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1 & -4 & 0\\ 2 & 0 & -1\end{bmatrix}$. Furthermore,

$$\begin{aligned}\det\left(\partial_F f\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right) &= \det\left(\begin{bmatrix}2 & 3\\ -6 & 1\end{bmatrix}\right)\\ &= 2+18\\ &= 20\\ &\neq 0\end{aligned}$$

so that $\partial_F f\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\in \mathrm{Isom}(\mathbb{R}^2;\mathbb{R}^2)$. In terms of 1.8, we have $U=\mathbb{R}^5$, and all the requirements to apply 1.8 are fulfilled, so that we conclude that $\exists V\in \mathrm{Open}(\mathbb{R}^3\times\mathbb{R}^2)$ such that $\left(\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\in V\subseteq U=\mathbb{R}^3\times\mathbb{R}^2$ and $\exists W\in\mathrm{Open}(\mathbb{R}^3)$ such that $\begin{bmatrix}3\\2\\7\end{bmatrix}\in W$ and $\exists g\in C^1(W;\mathbb{R}^2)$ such that

$$\left[\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix},\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\in V\wedge f\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix},\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=0\right]\boxed{\Leftrightarrow}\left[\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\in W\wedge\begin{bmatrix}x_1\\x_2\end{bmatrix}=g\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)\right]$$

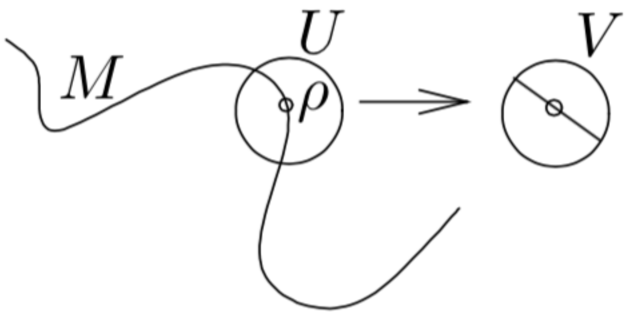
and we even can compute

$$\begin{aligned} g'\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) &= \begin{bmatrix} \partial_{y_1} g_1\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) & \partial_{y_2} g_1\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) & \partial_{y_3} g_1\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) \\ \partial_{y_1} g_2\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) & \partial_{y_2} g_2\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) & \partial_{y_3} g_2\left(\begin{bmatrix}3\\2\\7\end{bmatrix}\right) \end{bmatrix} \\ &= -\left[(\partial_F f)\left(\begin{bmatrix}3\\2\\7\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)\right]^{-1} (\partial_E f)\left(\begin{bmatrix}3\\2\\7\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= -\begin{bmatrix}2 & 3\\-6 & 1\end{bmatrix}^{-1} \begin{bmatrix}1 & -4 & 0\\2 & 0 & -1\end{bmatrix} \\ &= -\frac{1}{20}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix} \begin{bmatrix}1 & -4 & 0\\2 & 0 & -1\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{4} & \frac{1}{5} & -\frac{3}{20}\\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10}\end{bmatrix} \end{aligned}$$

2. Submanifolds (Untermannigfaltigkeit)

2.1. **Definition.** Let $(n, m) \in \mathbb{N}\backslash\{0\}$ such that $m \leqslant n$ and let $k \in \mathbb{N} \cup \{0, \infty\}$. A subset $M \subseteq \mathbb{R}^n$ is called an m-dimensional C^k-submanifold, iff $\forall p \in M$:

(1) $\exists U \in \text{Open}(\mathbb{R}^n)$ such that $p \in U$
(2) $\exists V \in \text{Open}(\mathbb{R}^n)$ such that $\exists \varphi : U \to V$ such that φ is a C^k-diffeomorphism.
(3) $\varphi(U \cap M) = V \cap \left(\mathbb{R}^m \times \left\{\underbrace{(0, 0, \dots, 0)}_{n-m \text{ times}}\right\}\right)$



in this example, $n = 2$, $m = 1$.

This definition can fail in many different places.

2.2. *Claim.* Let $(n, m) \in \mathbb{N}\backslash\{0\}$ such that $m \leqslant n$ and let $k \in \mathbb{N} \cup \{0, \infty\}$ and let $M \subseteq \mathbb{R}^n$. Then the following two statements are equivalent:

- M is an m-dimensional C^k-submanifold of $\mathbb{R}^n$.
- $\forall p \in M \exists U \in \text{Open}(\mathbb{R}^n) : p \in U$ and $\exists f : C^k(U; \mathbb{R}^{n-m})$ such that:
  - $U \cap M = \{x \in U \mid f(x) = 0\} \equiv f^{-1}(\{0\})$.
  - $f'(x) : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective $\forall x \in U \cap M$.

2.3. **Example.** $M := S^n \subseteq \mathbb{R}^{n+1}$ is a submanifold.

Proof. $M = S^n \equiv \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \|x\|^2 - 1 = 0 \right\}$, so take $U = \mathbb{R}^{n+1}$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by $x \mapsto \|x\| - 1$. Verify that f is C^∞, and also $f'(x) = \begin{bmatrix}2x_1 & 2x_2 & \dots & 2x_{n+1}\end{bmatrix} : \mathbb{R}^{n+1} \to \mathbb{R}$ is surjective unless $x = 0$, but $0 \notin U \cap M = M = S^n$. □

2.4. **Example.** $M := \underbrace{\left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_3 = 0 \right\}}_{x-y \text{ plane}} \cup \underbrace{\left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = 0 \wedge x_2 = 0 \right\}}_{z \text{ axis}} \subseteq \mathbb{R}^3$ is not a manifold.

Proof. This thing cannot be a manifold because on the plane it would have $m = 2$ and on the axis it would have $m = 1$, but m is fixed in the definition. □

3. Hints for Homework Sheet Number Seven

3.1. **Question 1.** See 1.10.

3.2. **Question 2.** Define $F : Sym(n) \times R(n) \to Sym(n)$ by $F(X, Y) := Y^T A(0) Y - X$. Observe that

$$\begin{aligned} F(A(0), \mathbb{1}) &= A(0) - A(0) \\ &= 0 \end{aligned}$$

- Compute $\partial_Y F(X, Y) : R(n) \to Sym(n)$.
- Evalute it at the point where F is zero: $\partial_Y F(A(0), \mathbb{1}) = ?$.
- Show that $\partial_Y F(A(0), \mathbb{1}) \in Isom(R(n); Sym(n))$.
- Thus the requirements for the implicit function theorem are fulfilled and we may employ it.

3.3. **Question 5.**

- (a): Use 2.2 with $f(x_1, x_2) = x_1 x_2$. Is $f'(0)$ surjective? Is $0 \in f^{-1}(\{0\})$ at all? Proceed to show that no other f as required by 2.2 can exist. This can be done by assuming that such an f exists, then removing the origin. In the domain that would create 4-connected components whereas in the range it would create 2-connected componenets.
- (b): Same spiel. Find 4-connected components in the domain.

References

[1] H. Cartan. *Differential Calculus*. Houghton Mifflin Co, 1971.
[2] Walter Rudin. *Principles of Mathematical Analysis (International Series in Pure and Applied Mathematics)*. McGraw-Hill Science/Engineering/Math, 1976.