Riemann's second proof of the analytic continuation of the Riemann Zeta function

Andreas Steiger

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1 Abstract

The Riemann zeta-function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $\operatorname{Re} s > 1$, but it is well known that there exists an analytic continuation onto the whole s-plane with a simple pole at $s = 1$. We want do derive this result using tools like the theta function

$$\theta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \quad t > 0, \tag{2}$$

its Mellin transform

$$\int_0^\infty \theta(t) t^s \frac{\mathrm{d}t}{t}, \quad s \in \mathbb{C},$$

the Gamma function

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t}, \quad \operatorname{Re} s > 0, \tag{3}$$

and Fourier transformation.
This proof of the analytic continuation is known as the second Riemannian proof.

2 Some tools

2.1 The Gamma function

Remark: The Gamma function has a large variety of properties. Most of those we use are very well known, but we will provide all the proofs anyways.

Proposition 1: $\Gamma(s)$ satisfies the functional equation

$$\Gamma(s+1) = s\,\Gamma(s) \tag{4}$$

Proof:

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{\mathrm{d}t}{t} = -e^{-t} t^s \big|_0^\infty - \int_0^\infty -e^{-t}\, s\, t^s \frac{\mathrm{d}t}{t} = s\,\Gamma(s)$$

□

Corollary: $\Gamma(s)$ has an analytic continuation on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$.

Proof: Using (4) we can find values for $-1 < \operatorname{Re} s \leq 0$, except for $s = 0$. These values give us an analytic continuation for $\operatorname{Re} s > -1$ with a simple pole at $s = 0$. Of course we can repeat this step infintely many times, with poles showing up at $s = 0, -1, -2, \ldots$. □

Proposition 2: $\Gamma(s)$ satisfies the functional equation

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Proof: $\Gamma(s)$ has an other representation which equals (3):

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^s}{1 + \frac{s}{n}}$$

With this representation and the Euler sine product we get

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= -s\Gamma(s)\Gamma(-s) \\
&= \frac{-s}{-s \cdot s} \prod_{n=1}^{\infty} \frac{1}{1 - \frac{s^2}{n^2}} \\
&= \frac{1}{s} \cdot \frac{\pi s}{\sin \pi s} \\
&= \frac{\pi}{\sin \pi s}
\end{aligned}$$

□

Corollary: $\dfrac{1}{\Gamma(s)} = \dfrac{\sin \pi s}{\pi}\,\Gamma(1-s)$ is entire.

Proof: $\Gamma(1-s)$ has simple poles at $s = 1, 2, \ldots$, but there $\sin \pi s$ has simple zeros, such that we get removable singularities. □

2.2 The Mellin transform

Definition: Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ be continuous. The Mellin transform $g(s)$ of f is defined by

$$g(s) := \int_0^\infty f(t) t^s \frac{\mathrm{d}t}{t}$$

for values s such that the integral converges.

Example: The Mellin transform of e^{-t} is $\Gamma(s)$.

Example:

$$f(t) = e^{-ct} \Rightarrow g(s) = c^{-s}\,\Gamma(s) \tag{5}$$

2.3 Fourier transform

Definition: Let $\mathcal{S}$ be the vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\pm\infty} |x|^n f(x) \to 0 \,\forall n \in \mathbb{N}$. For any $f \in \mathcal{S}$ we define the Fourier transform

$$\hat{f}(y) := \int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)\mathrm{d}x$$

and the integral converges for all $y \in \mathbb{C}$, $f \in \mathcal{S}$.

Example: Let $f(x) := e^{-\pi x^2}$, then $\hat{f} = f$.

Proof: Differentiating under the integral sign gives

$$\hat{f}'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)\mathrm{d}x = 2\pi i \int_{\infty}^{\infty} e^{-2\pi ixy} xe^{-\pi x^2}\mathrm{d}x$$

Integrating by parts yields

$$\begin{aligned}\hat{f}'(y) &= -2\pi ie^{-2\pi ixy}\frac{1}{-2\pi}e^{-\pi x^2}\Big|_{-\infty}^{\infty} + 2\pi i\int_{-\infty}^{\infty} 2\pi iye^{-2\pi ixy}\frac{e^{-\pi x^2}}{-2\pi}\mathrm{d}x \\ &= 2\pi y\int_{-\infty}^{\infty} 2^{-2\pi ixy} f(x)\mathrm{d}x = -2\pi y\hat{f}(y)\end{aligned}$$

Thus we have the differential equation

$$\frac{\hat{f}'(y)}{\hat{f}\,(y)} = -2\pi,$$

with the solution $\hat{f}(y) = Ce^{-\pi y^2}$. Setting $y = 0$ gives

$$C = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\mathrm{d}x = 1$$

and thus $\hat{f}(y) = e^{-\pi y^2} = f(y)$.

Lemma: Let $f \in \mathcal{S}$ and $g(x) := f(ax)$ for some $a > 0$. Then $\hat{g}(y) = \frac{1}{a}\hat{f}(\frac{1}{a})$.

Proof:

$$\begin{aligned}\hat{g}(y) &= \int_{-\infty}^{\infty} e^{-2\pi ixy} f(ax)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-2\pi i\frac{x}{a}y} f(x)\frac{\mathrm{d}x}{a} \\ &= \frac{1}{a}\hat{f}(\frac{y}{a})\end{aligned}$$

□

Proposition (Poisson Summation): If $g \in \mathcal{S}$, then

$$\sum_{m=-\infty}^{\infty} g(m) = \sum_{m=-\infty}^{\infty} \hat{g}(m).$$

Proof: Define $h(x) := \sum_{k=-\infty}^{\infty} g(x+k)$. Clearly this has period 1. Write down the Fourier series:

$$h(x) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m x}$$

with

$$\begin{aligned}
c_m &:= \int_0^1 h(x) e^{-2\pi i m x} \mathrm{d}x \\
&= \int_0^1 \sum_{k=-\infty}^{\infty} g(x+k) e^{-2\pi i m x} \mathrm{d}x \\
&= \sum_{k=-\infty}^{\infty} \int_0^1 g(x+k) e^{-2\pi i m x} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} g(x) e^{-2\pi i m x} \mathrm{d}x \\
&= \hat{g}(m)
\end{aligned}$$

Then

$$\sum_{k=-\infty}^{\infty} g(k) = h(0) = \sum_{m=-\infty}^{\infty} c_m e^{-2\pi i m \cdot 0} = \sum_{m=-\infty}^{\infty} c_m = \sum_{m=-\infty}^{\infty} \hat{g}(m)$$

□

2.4 The theta function

Remark: Note that $\theta(t)$ can not only be written as in (2), but also as

$$\theta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

We will often make use of that fact.

Proposition 4: $\theta(t)$ satisfies the functional equation

$$\theta(t) = \frac{1}{\sqrt{t}} \theta(1/t)$$

Proof: Let $g(x) := e^{-\pi t x^2}$ for a fixed $t > 0$ and $f(x) := e^{-\pi x^2}$. Obviously $g(x) = f(\sqrt{t}x)$. Using the lemma and the example from the Fourier transform we get

$$\hat{g}(y) = \frac{1}{\sqrt{t}}\hat{f}(y/\sqrt{t}) = \frac{1}{\sqrt{t}}f(y/\sqrt{t}) = \frac{1}{\sqrt{t}}e^{-\pi y^2/t}$$

Using Poisson summation we finally find that

$$\begin{aligned}\theta(t) &:= \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \sum_{n=-\infty}^{\infty} g(n) = \sum_{n=-\infty}^{\infty} \hat{g}(n) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} \\ &= \frac{1}{\sqrt{t}}\theta(1/t)\end{aligned}$$

□

Remark: In this proof we only need $\theta(t)$ for real-valued $t > 0$. But actually can also be looked at as a complex function for $\mathrm{Re}\, t > 0$ by analytic continuation. The functional equation still holds.

Proposition 5: As t goes to zero from above,

$$\left|\theta(t) - \frac{1}{\sqrt{t}}\right| < e^{-C/t} \qquad \text{for some } C > 0.$$

Proof: Using Proposition 4 and a rewrite of θ gives

$$\left|\theta(t) - \frac{1}{\sqrt{t}}\right| = \left|\frac{1}{\sqrt{t}}\left(\theta(1/t) - 1\right)\right| = \frac{1}{\sqrt{t}} \cdot 2\sum_{n=1}^{\infty} e^{-\pi n^2/t}$$

Suppose t is small enough such that $\sqrt{t} > 4 \cdot e^{-1/t}$ and $e^{-3\pi/t} < 1/2$. Then

$$\begin{aligned}\left|\theta(t) - \frac{1}{\sqrt{t}}\right| &< \frac{1}{2}e^{1/t}\left(e^{-\pi/t} + e^{-4\pi/t} + \cdots\right) \\ &< \frac{1}{2}e^{-(\pi-1)/t}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \\ &= e^{-(\pi-1)/t}\end{aligned}$$

and we see that $C = \pi - 1$ satisfies the inequality. □

3 The Theorem and its proof

Theorem $\zeta(s)$, as defined by (1) extends analytically onto the whole s-plane, except a simple pole at $s = 1$. Let

$$\Lambda(s) := \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s).$$

Then

$$\Lambda(s) = \Lambda(1 - s).$$

Proof: We want to consider the Mellin transform of the theta function,

$$\int_0^\infty \theta(t) t^s \frac{\mathrm{d}t}{t}$$

As t goes to infinity, $\theta(t)$ converges rapidly against 1, as all terms of the sum fall to zero rapidly, except for $n = 0$. By proposition 5 we see that for small t, $\theta(t)$ behaves like $t^{-1/2}$. Thus, if we want convergence at both ends, we have to introduce correction terms and define

$$\phi(s) := \int_1^\infty (\theta(t) - 1) t^{s/2} \frac{\mathrm{d}t}{t} + \int_0^1 \left(\theta(t) - \frac{1}{\sqrt{t}}\right) t^{s/2} \frac{\mathrm{d}t}{t}.$$

Note that we replaced s by $s/2$ in order to get $\zeta(s)$ instead of $\zeta(2s)$.
In the first integral, $\theta(t) - 1 \to 0$ extremely fast, such that the integral can be evaluated term by term, for any $s \in \mathbb{C}$. Similarly, $\theta(t) - \frac{1}{\sqrt{t}}$ is bounded above in the interval $(0, 1]$, such that the second integral converges for any $s \in \mathbb{C}$. These two statements together show that $\phi(s)$ is well-defined for all $s \in \mathbb{C}$, and it even is an entire function.
We now evalute the second integral, assuming $\operatorname{Re} s > 1$:

$$\begin{aligned}
& \int_0^1 \theta(t) t^{s/2} \frac{\mathrm{d}t}{t} - \int_0^1 t^{(s-1)/2} \frac{\mathrm{d}t}{t} \\
= & \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t} - \frac{2}{s-1} \\
= & \int_0^1 t^{s/2} \frac{\mathrm{d}t}{t} + 2 \int_0^1 \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t} + \frac{2}{1-s} \\
= & 2 \int_0^1 \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s}
\end{aligned}$$

Thus

$$\begin{aligned}
\phi(s) &= 2 \int_1^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t} + 2 \int_0^1 \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s} \\
&= 2 \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s}
\end{aligned}$$

for $\operatorname{Re} s > 1$.
Using (5) we can evaluate the integral and we get

$$\begin{aligned}
\frac{1}{2}\phi(s) &= \sum_{n=1}^{\infty} (\pi n^2)^{-s/2} \Gamma(\frac{s}{2}) + \frac{1}{s} + \frac{1}{1-s} \\
&= \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s) + \frac{1}{s} + \frac{1}{1-s} \\
\Rightarrow \zeta(s) &= \frac{\pi^{s/2}}{\Gamma(\frac{s}{2})} \left(\frac{1}{2}\phi(s) - \frac{1}{s} - \frac{1}{1-s} \right)
\end{aligned}$$

still for $\operatorname{Re} s > 1$. The only possible poles on the right hand side of the last equation are at $s = 0$ and $s = 1$, as $1/\Gamma(s)$ and $\phi(s)$ are entire functions. But for $s = 0$, the pole is removable, as the term causing the trouble is in fact

$$\frac{\pi^{s/2}}{\Gamma(\frac{s}{2})}\frac{1}{s} = \frac{\pi^{s/2}}{2\cdot\frac{s}{2}\Gamma(\frac{s}{2})} = \frac{\pi^{s/2}}{2\Gamma(\frac{s}{2}+1)} \overset{s\to 1}{\longrightarrow} \frac{1}{2\Gamma\frac{3}{2}}.$$

Thus we have found a meromorphic function on $\mathbb{C}$ with a (simple) pole at $s = 1$, and which equals $\zeta(s)$ for $\operatorname{Re} s > 1$. This is our analytic continuation.
It remains to prove that the functional equation

$$\Lambda(s) = \Lambda(1-s)$$

holds. Using

$$\frac{1}{2}\phi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) + \frac{1}{s} + \frac{1}{1-s} = \Lambda(s) + \frac{1}{s} + \frac{1}{s-1}$$

we have

$$\begin{aligned} \Lambda(s) &= \frac{1}{2}\phi(s) - \frac{1}{s} - \frac{1}{1-s} \\ \Lambda(1-s) &= \frac{1}{2}\phi(1-s) - \frac{1}{1-s} - \frac{1}{s} \end{aligned}$$

and we only have to prove $\phi(s) = \phi(1-s)$:

$$\begin{aligned} \phi(s) &= \int_1^\infty (\theta(t)-1)t^{s/2}\frac{\mathrm{d}t}{t} + \int_0^1 \left(\theta(t) - \frac{1}{\sqrt{t}}\right) t^{s/2}\frac{\mathrm{d}t}{t} \\ &\overset{t\to\frac{1}{t}}{=} \int_0^1 \left(\theta\left(\frac{1}{t}\right) - 1\right) t^{-s/2}\frac{\mathrm{d}t}{t} + \int_1^\infty \left(\theta\left(\frac{1}{t}\right) - \sqrt{t}\right) t^{-s/2}\frac{\mathrm{d}t}{t} \\ &= \int_0^1 (\sqrt{t}\theta(t)-1)t^{-s/2}\frac{\mathrm{d}t}{t} + \int_1^\infty \left(\sqrt{t}\theta(t) - \sqrt{t}\right) t^{-s/2}\frac{\mathrm{d}t}{t} \\ &= \int_0^1 \left(\theta(t) - \frac{1}{\sqrt{t}}\right) t^{(1-s)/2}\frac{\mathrm{d}t}{t} + \int_1^\infty (\theta(t)-1)\, t^{(1-s)/2}\frac{\mathrm{d}t}{t} \\ &= \phi(1-s) \end{aligned}$$

□