Q1 Let $B \subseteq \mathbb{R}^n$ be a bounded set and let $h \in (0,\infty) \subseteq \mathbb{R}$.
The cone over B of height h is defined as:

$$K_h(B) := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; y \in [0,h] \wedge x \in \left(1-\tfrac{y}{h}\right)B \right\}$$

Claim: $\partial K_h(B) = K_h(\partial B) \cup (B \times \{0\})$

Proof: Claim: $\overline{K_h(B)} = K_h(\bar{B})$

Proof: $\boxed{\supseteq}$ Let $(x,y) \in K_h(\bar{B})$ be given.

Let $\varepsilon > 0$ be given. Goal: $B_\varepsilon((x,y)) \cap K_h(B) \neq \emptyset$.

$(x,y) \in K_h(\bar{B})$ means $y \in [0,h] \wedge x \in (1-\frac{y}{h})\bar{B}$

Case 1: $y = h$

Then $x \in 0\bar{B} = \{0\}$. $\Rightarrow x = 0$

Then $(0,h) \in K_h(B) \subseteq \overline{K_h(B)}$ ✓

Case 2: $y \in [0,h)$

$x \in (1-\frac{y}{h})\bar{B} \iff \frac{h}{h-y}x \in \bar{B}$

$\Rightarrow B_\delta(\frac{h}{h-y}x) \cap B \neq \emptyset$

Pick $\delta := \frac{h}{h-y}\varepsilon$.

$\Rightarrow \exists\, \tilde{x} \in B$ s.t. $\|\tilde{x} - \frac{h}{h-y}x\| < \frac{h}{h-y}\varepsilon$

Claim: $(\frac{h-y}{h}\tilde{x}, y) \in B_\varepsilon((x,y)) \cap K_h(B)$

Proof: $\|(\frac{h-y}{h}\tilde{x}, y) - (x,y)\| = \|\frac{h-y}{h}\tilde{x} - x\|$

$= \frac{h-y}{h}\|\tilde{x} - \frac{h}{h-y}x\| < \varepsilon$ ✓

$y \in [0,h) \subseteq [0,h]$ ✓

$\tilde{x} \in B \Rightarrow \frac{h-y}{h}\tilde{x} \in \frac{h-y}{h}B = (1-\frac{y}{h})B$ ✓

$\boxed{\subseteq}$ Claim: $K_h(\bar{B}) \supseteq K_h(B)$

Proof: Let $(x,y) \in K_h(B)$ be given. Then

$y \in [0,h] \wedge x \in (1-\frac{y}{h})B$

But $B \subseteq \bar{B} \Rightarrow (1-\frac{y}{h})B \subseteq (1-\frac{y}{h})\bar{B}$. $\Rightarrow x \in (1-\frac{y}{h})\bar{B}$.

$\Rightarrow (x,y) \in K_h(\bar{B})$

Claim: $K_h(\bar{B}) \in \mathrm{Closed}(\mathbb{R}^n \times \mathbb{R})$

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Proof: Define $f : \bar{B} \times [0,h] \to \mathbb{R}^{n+1}$ by

$$(x,y) \longmapsto ((1-\tfrac{y}{h})x, y)$$

Claim: $f(\bar{B}\times[0,h]) = K_h(\bar{B})$

Claim: f is cont.

Claim: The images of cpt. sets under cont. maps are cpt.

Proof: Let $\{V_\alpha\}_{\alpha\in A}$ be some open cover of $f(X)$, where X is cpt. and $f: X \to Y$ is cont.

Then as $f \in C(X,Y)$, $f^{-1}(V_\alpha) \in \mathrm{Open}(X)$ $\forall \alpha \in A$.

Observe that $\{f^{-1}(V_\alpha)\}_{\alpha\in A}$ covers X (since every point in X must go somewhere in $f(X)$). But X is cpt.

$\Rightarrow \exists \{\alpha_i\}_{i=1}^{s} \subseteq A$ s.t. $\{f^{-1}(V_{\alpha_i})\}_{i=1}^{s}$ covers X.

$$\bigcup_{i\in J_s} f^{-1}(V_{\alpha_i}) \supseteq X$$

$$\Rightarrow \underbrace{f(\bigcup_{i\in J_s} f^{-1}(V_{\alpha_i}))}_{\underbrace{f(f^{-1}(\bigcup_{i\in J_s} V_{\alpha_i}))}_{\subseteq \bigcup_{i\in J_s} V_{\alpha_i}}} \supseteq f(X)$$

Claim: $\bar{B}\times[0,h]$ is cpt.

Proof: It is bounded, and it is closed bcs. the product of closed sets is closed: If $M^{\subseteq X}$ and $L^{\subseteq Y}$ are closed, M^c and L^c are open.

$$(M\times L)^c = \underbrace{(M^c\times Y)}_{\text{open}} \cup \underbrace{(X\times L^c)}_{\text{open}}$$

$\Longrightarrow M\times L$ is closed.

As $K_h(\bar{B})$ is cpt., it is also closed.

But as $\overline{K_h(\mathring{B})}$ is closed and contains $K_h(B)$,

it must contain $\overline{K_h(B)}$.

Define $U_h(B) :\!= \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid y \in (0,h) \wedge x \in (1-\frac{y}{h})\mathring{B}\}$

Claim: $\mathring{K_h}(B) = U_h(B)$

Proof: $\boxed{\supseteq}$ Claim: $U_h(B) \in \mathrm{Open}(\mathbb{R}^{n+1})$

Proof: Define $f: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$(x,y) \longmapsto \left(\frac{h}{h-y} x, y\right)$$

Then f is cont., and

$$f^{-1}(\mathring{B} \times (0,h)) \equiv \{(x,y) \in \mathbb{R}^{n+1} \mid \tfrac{h}{h-y} x \in \mathring{B} \wedge y \in (0,h)\}$$
$$= \{(x,y) \in \mathbb{R}^{n+1} \mid x \in (1-\tfrac{y}{h})\mathring{B} \wedge y \in (0,h)\}$$
$$\equiv U_h(B)$$

But $\mathring{B} \times (0,h)$ is open.

Claim: $U_h(B) \subseteq K_h(B)$

Proof: If $(x,y) \in U_h(B)$ then $y \in (0,h)$ and $x \in (1-\frac{y}{h})\mathring{B}$.

But $(0,h) \subseteq [0,h]$ and $\mathring{B} \subseteq B$, so that

$$(1-\tfrac{y}{h})\mathring{B} \subseteq (1-\tfrac{y}{h})B \quad \forall\, y \in (0,h).$$

As an open set inside $K_h(B)$, $U_h(B)$ thus must be in the interior of $K_h(B)$.

$\boxed{\subseteq}$ Let $(x,y) \in \mathring{K_h}(B)$ be given. If $(x,y) \in U_h(B)$ we're finished, so assume otherwise.

But $\mathring{K_h}(B) \subseteq K_h(B)$, so that $(x,y) \in K_h(B)$.

$$\Rightarrow \left[y \in [0,h] \wedge x \in (1-\tfrac{y}{h})B\right] \wedge \left[\neg\left[y \in (0,h) \wedge x \in (1-\tfrac{y}{h})\mathring{B}\right]\right]$$
$$\Leftrightarrow \left[y \in [0,h] \wedge x \in (1-\tfrac{y}{h})B\right] \wedge \left[y \notin (0,h) \vee x \notin (1-\tfrac{y}{h})\mathring{B}\right]$$
$$\Rightarrow \left[y \in \{0,h\} \wedge x \in (1-\tfrac{y}{h})B\right] \vee \left[y \in [0,h] \wedge x \in (1-\tfrac{y}{h})B \setminus (1-\tfrac{y}{h})\mathring{B}\right]$$

Case 1: $y \in \{0,h\} \wedge x \in (1-\frac{y}{h})B$

We know that $(x,y) \in \mathring{K_h}(B)$, $\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon((x,y)) \subseteq K_h(B)$. But this is impossible because then $\exists$ some $(\tilde{x},\tilde{y}) \in K_h(B)$ with $\tilde{y} \notin [0,h]$.

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Case 2: $y\in[0,h] \wedge x\in\left(\left(1-\frac{y}{h}\right)B\right)\setminus\left(\left(1-\frac{y}{h}\right)\mathring{B}\right)$

As we've already seen, it's impossible that $y\in\{0,h\}$ so that we assume $y\in(0,h)$.

$$\left[\left(1-\frac{y}{h}\right)B\right]\setminus\left[\left(1-\frac{y}{h}\right)\mathring{B}\right] =$$

$$\left\{\left(1-\frac{y}{h}\right)x \,\middle|\, x\in B\right\}\setminus\left\{\left(1-\frac{y}{h}\right)x \,\middle|\, x\in\mathring{B}\right\} =$$

$$=\left(1-\frac{y}{h}\right)\left(B\setminus\mathring{B}\right)$$

Bcs. $(x,y)\in\mathring{K}_h(B)$, $\exists\ \varepsilon_0>0$ s.t. $B_{\varepsilon_0}(x,y)\subseteq K_h(B)$

But $x\notin\left(1-\frac{y}{h}\right)\mathring{B} \Rightarrow \forall\,\delta>0,\ B_\delta(x)\not\subseteq\left(1-\frac{y}{h}\right)B$

$\Rightarrow \forall\,\delta>0,\ \exists\ x_\delta\in\mathbb{R}^n$ s.t. $\|x-x_\delta\|<\delta$ yet $x_\delta\notin\left(1-\frac{y}{h}\right)B$.

$\Rightarrow \exists\ X_{\varepsilon_0}\in\mathbb{R}^n$ s.t. $\|x-x_{\varepsilon_0}\|<\varepsilon_0 \wedge x_{\varepsilon_0}\notin\left(1-\frac{y}{h}\right)B$.

$\Rightarrow \exists\ (x_\varepsilon,y)\in\mathbb{R}^{n+1}$ s.t. $\|(x,y)-(x_\varepsilon,y)\|<\varepsilon \wedge (x_\varepsilon,y)\notin K_h(B)$

$\Rightarrow B_{\varepsilon_0}((x,y))\not\subseteq K_h(B) \Rightarrow \boxed{\perp}$

$$\Rightarrow \partial K_h(B) \equiv \overline{K_h(B)}\setminus\mathring{K}_h(B)$$

$$= K_h(\overline{B})\setminus\mathcal{U}_h(B)$$

$$\equiv\left\{(x,y)\in\mathbb{R}^{n+1}\,\middle|\, y\in[0,h]\wedge x\in\left(1-\tfrac{y}{h}\right)\overline{B}\right\}\setminus$$
$$\left\{(x,y)\in\mathbb{R}^{n+1}\,\middle|\, y\in(0,h)\wedge x\in\left(1-\tfrac{y}{h}\right)\mathring{B}\right\}$$

$$=\left\{(x,y)\in\mathbb{R}^{n+1}\,\middle|\,\left[y\in[0,h]\wedge x\in\left(1-\tfrac{y}{h}\right)\overline{B}\right]\wedge\,!\underbrace{\left[y\in(0,h)\wedge x\in\left(1-\tfrac{y}{h}\right)\mathring{B}\right]}_{y\notin(0,h)\,\vee\, x\notin(1-\frac{y}{h})\mathring{B}}\right\}$$

$$\underbrace{\left[y\in\{0,h\}\wedge x\in\left(1-\tfrac{y}{h}\right)\overline{B}\right]}_{(y=0\,\wedge\, x\in\overline{B})\,\vee\,(y=h\,\wedge\, x=0)}\vee\underbrace{\left[y\in[0,h]\wedge x\in\left(1-\tfrac{y}{h}\right)\overline{B}\setminus\mathring{B}\right]}_{y\in[0,h]\,\wedge\,\in(1-\frac{y}{h})\partial B}$$

$$= K_h(\partial B)\cup\left(\overline{B}\times\{0\}\right)\cup\left(\{(0,h)\}\right)$$

$\partial B\times\{0\}$ already contains that boundary.

$$= K_h(\partial B)\cup\overline{B}\times\{0\}$$

$$= K_h(\partial B)\cup B\times\{0\}$$

Q2 (a) <u>Claim</u>: $\exists\ \mathcal{U}\in \mathrm{Open}(\mathbb{R})$ s.t. $\partial\mathcal{U}$ has non-zero Jordan content.

<u>Proof</u>: Let $\{a_i\}_{i\in\mathbb{N}}$ be an enumeration of the rationals in $[0,1]$.
(exists bcs. $\mathbb{Q}$ is countable!)

Choose any $\varepsilon\in(0,1/2)$.

$\forall\ n\in\mathbb{N}$ define $I_n := \left(a_n - \varepsilon 2^{-(n+1)},\ a_n+\varepsilon 2^{-(n+1)}\right)$

and $\mathcal{U} := \bigcup_{n\in\mathbb{N}} I_n$.

Then $\mathcal{U}\in \mathrm{Open}(\mathbb{R})$ ✓

<u>Claim</u>: $\left([0,1]\setminus\mathcal{U}\right)\subseteq\partial\mathcal{U}$

<u>Proof</u>: <u>Claim</u>: $[0,1]\subseteq\overline{\mathcal{U}}$

<u>Proof</u>: First note that $\{0,1\}\subseteq\mathcal{U}\subseteq\overline{\mathcal{U}}$, as they are rational, and $\forall n\in\mathbb{N}$, $a_n\in I_n\subseteq\mathcal{U}$.

Next, for $(0,1)\subseteq\overline{\mathcal{U}}$:

$(0,1)\setminus\overline{\mathcal{U}} = \underbrace{(0,1)}_{\text{open}}\cap\underbrace{(\mathbb{R}\setminus\overline{\mathcal{U}})}_{\text{open}}\in \mathrm{Open}$

Assume $(0,1)\setminus\overline{\mathcal{U}}\neq\emptyset$. $\Rightarrow\exists\ (a,b)\subseteq(0,1)\setminus\overline{\mathcal{U}}$.

But every interval in $(0,1)$ would contain a rational point, which would be in $\mathcal{U}\subseteq\overline{\mathcal{U}}$.

$\Rightarrow [0,1]\setminus\mathcal{U}\subseteq\overline{\mathcal{U}}\setminus\mathcal{U}\equiv\partial\mathcal{U}$

<u>Claim</u>: $\partial\mathcal{U}$ has non-zero Jordan content.

<u>Proof</u>: Assume otherwise. $\Rightarrow$ for our <u>same</u> $\varepsilon\in(0,1/2)$,

$\exists\ \{J_i\}_{i=1}^m$ where J_i is an open interval s.t.

$\partial\mathcal{U}\subseteq\bigcup_{i=1}^m J_i$ and $\left(\sum_{i=1}^m |J_i|\right)<\varepsilon$.

But $\overline{\mathcal{U}}\equiv\mathcal{U}\cup\partial\mathcal{U}$

$\Rightarrow [0,1]\subseteq\overline{\mathcal{U}} = \mathcal{U}\cup\partial\mathcal{U}\subseteq\left(\bigcup_{n\in\mathbb{N}} I_n\right)\cup\left(\bigcup_{i=1}^m J_i\right)$

which is an open cover of the compact set $[0,1]$.

$\Rightarrow\exists\ N\in\mathbb{N}$ s.t. $[0,1]\subseteq\left(\bigcup_{n=1}^N I_n\right)\cup\left(\bigcup_{i=1}^m J_i\right)$

But if $A\subseteq B$ then $\mathrm{vol}(A)\leq\mathrm{vol}(B)$.

$$\Rightarrow \mathrm{vol}([0,1]) \leq \mathrm{vol}\left(\bigcup_{i=1}^{m} J_i \cup \bigcup_{n=1}^{N} I_n\right)$$
$$\leq \sum_{i=1}^{m} \mathrm{vol}(J_i) + \sum_{n=1}^{N} \mathrm{vol}(I_n)$$
$$\leq \varepsilon + \varepsilon \sum_{n=1}^{N} 2^{-n}$$
$$\leq 2\varepsilon$$
$$< 1$$

$\boxed{\text{But}}$ $\mathrm{vol}([0,1]) = 1. \Rightarrow 1 < 1 \Rightarrow \boxed{\perp}$ ∎

b) Claim: $\exists\, W \in \mathrm{Open}(\mathbb{R}^2) \cap \mathrm{Connected}(\mathbb{R}^2)$ s.t. ∂W has non zero Jordan content.

Proof: Define $W := (\mathcal{U} \times [0,1)) \cup ((-1,2) \times (-1,0))$

where $\mathcal{U}$ is as in part (a)

Claim: $W \in \mathrm{Open}(\mathbb{R}^2)$

Proof: Claim: $\mathcal{U} \subseteq (-1,2)$ by its def.

Then if $x \in \mathcal{U} \times \{0\}$, then we can always find a ball around it in W.

Claim: $[0,1]^2 \subseteq \overline{W} \left(= \overline{\mathcal{U} \times [0,1)} \cup \overline{(-1,2) \times (-1,0)}\right) = \overline{\mathcal{U}} \times \overline{[0,1)} \cup \overline{(-1,2)} \times \overline{(-1,0)}$

Proof: Let $(x,y) \in [0,1]^2$ be given.

Then $x \in [0,1] \subseteq \overline{\mathcal{U}}$ by part (a).

$y \in [0,1] \in \overline{[0,1)}$

Claim: W is connected.

Proof: W looks like a comb, and is thus path-connected.

Recall that every path-connected space is connected.

Claim: ∂W has non zero Jordan content.

Proof: Assume otherwise. Pick any $\varepsilon \in (0, 1/3)$.

Then $\exists$ 2-cells $\{J_i\}_{i=1}^{m}$ s.t.

$\partial W \subseteq \bigcup_{i=1}^{m} J_i$ and $\left(\sum_{i=1}^{m} vol_2(J_i)\right) < \varepsilon$

Define $\forall n \in \mathbb{N}$, $\tilde{I}_n := I_n \times (-1/2, 3/2)$.

Claim: $\{\tilde{I}_n\}_{n\in\mathbb{N}}$ and $\{J_i\}_{i=1}^{m}$ cover $[0,1]^2$.

Proof: Let $(x,y) \in [0,1]^2$ be given.

$[0,1]^2 \subseteq \overline{W} = W \cup \partial W$

$\implies$ If $(x,y) \in \partial W$ we are finished.

Otherwise, if $(x,y) \in W \cap [0,1]^2$, then $y \in (-1/2, 3/2)$ indeed and $x \in \mathcal{U} \subseteq \bigcup_{n\in\mathbb{N}} I_n$.

But $[0,1]^2$ is compact. $\implies \exists N \in \mathbb{N}$ s.t.

$[0,1]^2 \subseteq \bigcup_{n=1}^{N} \tilde{I}_n \cup \bigcup_{j=1}^{m} J_i$

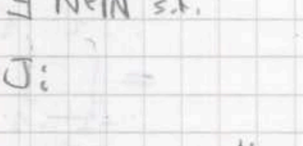
$\implies vol_2([0,1]^2) \leq \sum_{n=1}^{N} vol_2(\tilde{I}_n) + \sum_{i=1}^{N} vol_2(J_i)$

$\leq 2\varepsilon + \varepsilon$

< 1

But $vol_2([0,1]^2) = 1$. $\implies 1 < 1 \implies \boxed{\bot}$

Q3 Let $B \subseteq \mathbb{R}^n$ be an arbitrary subset and $K \subseteq \mathbb{R}^n$ be convex s.t. $(K \not\subseteq B) \wedge [(K \cap B) \neq \emptyset]$.



Claim: $(K \cap \partial B) \neq \emptyset$

Proof: Define $K_1 := K \cap (\mathbb{R}^n \setminus \overline{B})$

$K_2 := K \cap (\mathring{B})$

$\boxed{\overline{B} = \mathring{B} \cup \partial B = B \cup \partial B}$

$(K_1, K_2) \in [\mathrm{Open}(K)]^2$

$\overline{B}^c = (B \cup \partial B)^c = B^c \cap \partial B^c$

Assume $K \cap \partial B = \emptyset$. If $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$ then $\exists$ sep. of K.

Claim: Convex sets are connected.

Proof: Convex sets are path-connected $\implies$ connected.

Claim: $K_1 \cap K_2 = \emptyset$

Proof: $K_1 \cap K_2 = (K \cap (\mathbb{R}^n \setminus \overline{B})) \cap (K \cap \mathring{B}) = K \cap (\mathbb{R}^n \setminus \overline{B}) \cap \mathring{B}$

$= K \cap (\mathring{B} \setminus \overline{B}) = \emptyset$ bcs. $\overline{B} \supseteq \mathring{B}$.

Claim: $K_1 \cup K_2 = K$

Proof: $K = K \cap \mathbb{R}^n = K \cap \left[(\mathbb{R}^n \setminus \bar{B}) \cup \bar{B}\right]$
$= K \cap \left[(\mathbb{R}^n \setminus \bar{B}) \cup (\mathring{B} \cup \partial B)\right]$
$= \left[K \cap (\mathbb{R}^n \setminus \bar{B})\right] \cup \left[K \cap \mathring{B}\right] \cup \left[K \cap \partial B\right]$
$= K_1 \cup K_2 \cup \emptyset$

$\Longrightarrow \boxed{1}$

Q4 (a) $K = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

$$vol_2(K) \equiv \int_{I^2} \chi_K(x,y)\, dx\, dy$$

for any 2-cell I^2 which contains $\bar{K}$, assuming K is "nice" enough (that is, compact).

$$= \int_{-1}^{1}\int_{-1}^{1} \chi_K(x,y)\, dx\, dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 \; dx\, dy$$

$$= \int_{-1}^{1} 2\sqrt{1-y^2}\, dy$$

$y = \sin(t)$
$dy = \cos(t)\, dt$

$$= 2\int_{t=-\pi/2}^{\pi/2} \sqrt{1-\sin^2(t)}\, \cos(t)\, dt$$

$$= 2\int_{-\pi/2}^{\pi/2} \cos^2(t)\, dt = 2\int_{-\pi/2}^{\pi/2} \tfrac{1}{2}\left[1+\cos(2t)\right] dt$$

$$= \pi$$

(b) $R_f = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [a,b] \wedge x^2+y^2 \leq (f(z))^2\}$

where $f: [a,b] \to \mathbb{R}$ is cont.

Then $vol_3(R_f) = \int_{I^3} \chi_{R_f}(x,y,z)\, dx\, dy\, dz$

$$= \int_a^b \int_{-f(z)}^{f(z)} \int_{-\sqrt{f(z)^2-y^2}}^{\sqrt{f(z)^2-y^2}} 1 \; dx\, dy\, dz$$

$$= \int_a^b \pi (f(z))^2\, dz$$

Using the fact that $vol_n(\lambda B) = \lambda^n vol_n(B)$, $\lambda \geq 0$, B msrbl.

(c) $G = \left\{ x \in \mathbb{R}^4 \;\middle|\; x_1^2 + x_2^2 \leq x_3^2 + x_4^2 \leq 1 \right\}$

$$\mathrm{vol}_4(G) = \int_{[-1,1]^4} \chi_G(x)\, d^4x$$

$$= \int_{\{x_3^2+x_4^2 \leq 1\}} \int_{\{x_1^2+x_2^2 \leq x_3^2+x_4^2\}} 1 \; d^4x$$

$$= \int_{\{x_3^2+x_4^2 \leq 1\}} \pi(x_3^2+x_4^2)\, dx_3\, dx_4$$

$r = \sqrt{x_3^2+x_4^2}$

$\theta = \mathrm{arctg}\left(\frac{x_4}{x_3}\right)$

$$= \pi \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} r^2\, r\, dr\, d\theta$$

$$= 2\pi^2 \int_0^1 r^3\, dr = 2\pi^2 \frac{1}{4} = \boxed{\frac{\pi^2}{2}}$$