HW#11

The Transf. Formula

Let $\mathcal{U} \in \mathrm{Open}(\mathbb{R}^n)$ and $\varphi: \mathcal{U} \to \mathbb{R}^n$ be C^1.
Let $A \subseteq \mathcal{U}$ be a compact Jordan measurable subset and $N \subseteq A$ be a Jordan null set.
Assume $\varphi|_{A \setminus N}: A \setminus N \to \mathbb{R}^n$ is injective and

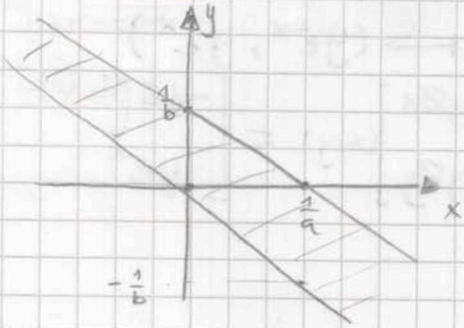
$\det(\varphi'(x)) \neq 0 \quad \forall \; x \in A \setminus N$.

Then $\varphi(A)$ is Jordan measurable, and if $f: \varphi(A) \to \mathbb{R}$ is Riemann integrable, so is $f \circ \varphi: A \to \mathbb{R}$ and

$$\boxed{\int_{\varphi(A)} f(x)\,dx = \int_A (f \circ \varphi)(x) \, |\det(\varphi'(x))| \, dx}$$

Q1 (a) Let $(a,b,c,d)\in\mathbb{R}^4$: $ad-bc\neq 0$.

Define $A_1 := \{(x,y)\in\mathbb{R}^2 \mid 0\le ax+by\le 1 \wedge 0\le cx+dy\le 1\}$



$ax+by=1 \iff y=-\frac{a}{b}x+\frac{1}{b}$

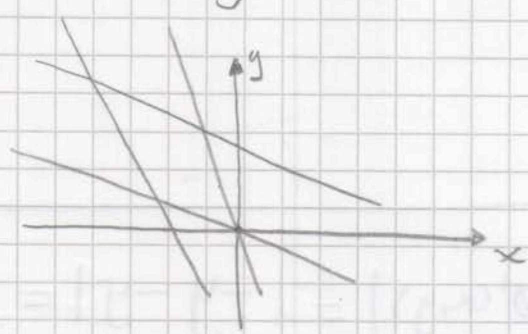
$ax+by=0$

If the line $ax+by=1$ is parallel to $cx+dy=1$ then

$-\frac{a}{b}=-\frac{c}{d} \iff ad-bc=0$

Thus the given condition exactly means the area A_1 is not infinite.

Then we have



$\emptyset(A_1)=[0,1]^2$

Define $\emptyset:\mathbb{R}^2\to\mathbb{R}^2$ by $(x,y)\longmapsto(ax+by,\ cx+dy)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Then $\emptyset$ is injective as $\ker(\emptyset)=\{(0,0)\}$, as $\det(\begin{bmatrix}a&b\\c&d\end{bmatrix})\neq 0$.

We also know $\emptyset'(x,y)=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, so that $\emptyset'$ is continuous (as it is a constant map) and thus $\emptyset\in C^1(\mathbb{R}^2,\mathbb{R}^2)$.

In addition we have $\det(\emptyset'(x,y))=\det(\begin{bmatrix}a&b\\c&d\end{bmatrix})\neq 0$.

Next, let χ_{A_1} be the characteristic function of A_1.

Then observe: $\chi_{A_1}(x,y)=(\chi_{[0,1]^2}\circ\emptyset)(x,y)$.

χ_{A_1} is Riemann integrable.

Apply the "Change of Variables" theorem with: $U=\mathbb{R}^2$, $A=\overline{A_1}$, $N=\emptyset$, $\varphi=\emptyset^{-1}$, $f=\chi_{A_1}$

$$\Longrightarrow \int_{\mathbb{R}^2}\chi_{A_1}=\int_{A_1}1=\int_{\emptyset^{-1}([0,1]^2)}(1\circ\emptyset^{-1})(x,y)\,|ad-bc|^{-1}\,dx\,dy = |ad-bc|^{-1}\int 1 = |ad-bc|$$

2 (b) Let $0<a<b$ and $0<c<d$ and define:

$$A_2 := \{(x,y)\in\mathbb{R}^2 \mid a \leq ye^{-x} \leq b \,,\ c \leq ye^{+x} \leq d\}$$

Then define $\emptyset : A_2 \to [a,b]\times[c,d]$ $\qquad \emptyset^{-1} : [a,b]\times[c,d] \to A_2$

$$(x,y) \longmapsto (ye^{-x}, ye^{x})$$

$$\emptyset'(x,y) = \begin{bmatrix} \partial_x \emptyset_x & \partial_y \emptyset_x \\ \partial_x \emptyset_y & \partial_y \emptyset_y \end{bmatrix}(x,y) = \begin{bmatrix} -ye^{-x} & e^{-x} \\ ye^{x} & e^{x} \end{bmatrix}$$

$\Longrightarrow \emptyset \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

<u>Claim</u>: $\emptyset$ is injective

<u>Proof</u>: Assume $(ye^{-x}, ye^{x}) = (\tilde{y}e^{-\tilde{x}}, \tilde{y}e^{\tilde{x}})$

$$\Longrightarrow \begin{cases} ye^{-x} = \tilde{y}e^{-\tilde{x}} & ① \\ ye^{x} = \tilde{y}e^{\tilde{x}} & ② \end{cases} \overset{①\cdot②}{\Longrightarrow} y^2 = \tilde{y}^2 \overset{①}{\Longrightarrow} y = \tilde{y}$$

$\Downarrow$

$x = \tilde{x}$ as exp is injective.

$$|\emptyset'(x,y)| = |-y-y| = 2|y| \overset{\substack{a>0 \\ c>0}}{=} 2y$$

$$\Longrightarrow vol(A_2) = \int_{\mathbb{R}^2} \chi_{A_2} = \int_{A_2} 1 = \int_{\emptyset^{-1}([a,b]\times[c,d])} 1$$

$$= \int_{[a,b]\times[c,d]} \left((1 \circ \emptyset^{-1})(x,y)\right) \left|\det(\emptyset^{-1\prime}(x,y))\right| dx dy$$

$$\det(\emptyset^{-1\prime}(\emptyset(x,y))) \underset{\substack{\text{Inverse function} \\ \text{theorem}}}{=} \det\left[(\emptyset'(x,y))^{-1}\right] = \frac{1}{2y}$$

$$\Longrightarrow \det(\emptyset^{-1\prime}(x,y)) = \frac{1}{2(\emptyset^{-1}(x,y))_y} = \frac{1}{2\sqrt{xy}}$$

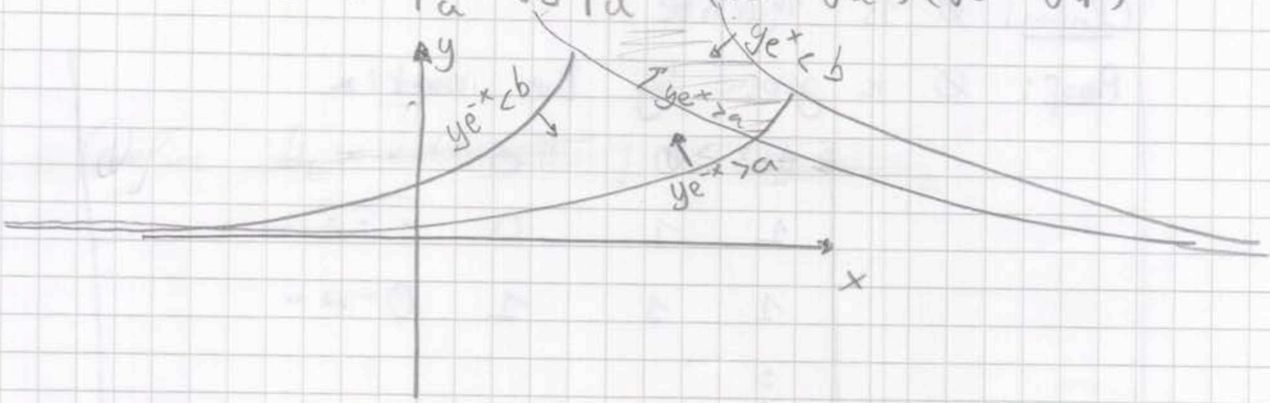
Observe $\emptyset^{-1} : (x,y) \longmapsto (\frac{1}{2}\log(\frac{y}{x}), \sqrt{xy})$ because

$$\emptyset \circ \emptyset^{-1} : (x,y) \longmapsto (\sqrt{xy}\exp(-\tfrac{1}{2}\log(\tfrac{y}{x})), \sqrt{xy}\exp(\tfrac{1}{2}\log(\tfrac{y}{x}))) = (x,y)$$

We continue with the integral to obtain:

$$= \int_{[a,b]\times[c,d]} \frac{1}{2\sqrt{xy}}\, dx\, dy = \frac{1}{2}\int_{[a,b]} \frac{1}{\sqrt{x}}\, dx \int_{[c,d]} \frac{1}{\sqrt{y}}\, dy$$

$$= \sqrt{x}\Big|_a^b \sqrt{y}\Big|_d^c = (\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{d})$$



(c) $A_3 := \left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; 0 \le x+y \le 1 \;\middle|\; \wedge \;\; 0 \le 2x-3y \le 4 \right\}$

Define $\phi : A_3 \to [0,1]\times[0,4]$ by $(x,y) \longmapsto (x+y, 2x-3y)$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rinv.

Then $\det(\phi') = -5 \neq 0$ and we have:

$$\int_{A_3} \sqrt{x+y}\, dx\, dy = \int_{\phi^{-1}([0,1]\times[0,4])} \sqrt{x+y}\, dx\, dy$$

$$= \int_{[0,1]\times[0,4]} \sqrt{(\phi^{-1}(x,y))_x + (\phi^{-1}(x,y))_y}\; \frac{1}{5}\, dx\, dy$$

$$\boxed{\phi^{-1} : (x,y) \longmapsto \begin{bmatrix} 3/5 & 1/5 \\ 2/5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}(3x+y,\, 2x-y)}$$

$$= \int_{[0,1]\times[0,4]} \sqrt{\frac{1}{5} 5x}\; \frac{1}{5}\, dx\, dy = \frac{4}{5}\int_0^1 \sqrt{x}\, dx = \frac{4}{5}\cdot\frac{2}{3} = \boxed{\frac{8}{15}}.$$

$\boxed{Q2}$ $\Delta^n := \left\{ x \in [0,1]^n \;\middle|\; \sum_{j=1}^{n} x_j \le 1 \right\} \subseteq \mathbb{R}^n$

is the n-dim. std. simplex.

4

(a) What is $vol_n(\Delta^n)$?

Define $D_n := \{ x \in \mathbb{R}^n \mid 0 \leq x_1 \leq \dots \leq x_n \leq 1 \}$

$$\varnothing : \Delta^n \to D_n \qquad x \longmapsto (x_1, x_1+x_2, \dots, x_1+x_2+\dots+x_n)$$

Claim: $\varnothing$ is well defined ✓

Claim: $\varnothing$ is injective

Proof: $\varnothing$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \\ 1 & 1 & 0 & \cdots & \\ 1 & 1 & 1 & 0 \cdots & \\ \vdots & & & & \\ 1 & 1 & 1 & 1 \cdots & 1 \end{bmatrix}$$

which has determinant 1 and so $ker(\varnothing) = \{0\}$.

$\Longrightarrow det(\varnothing') \neq 0$

$$\Longrightarrow vol_n(\Delta^n) = \int_{\Delta^n} 1 = \int_{\varnothing^{-1}(D_n)} 1$$

$$= \int_{D_n} \underbrace{(1 \circ \varnothing^{-1})(x)}_{1} \underbrace{|det(\varnothing^{-1\,\prime})(x)|}_{1} dx$$

$$= \int_{D_n} dx = \int_{x_n=0}^{x_n=1} \int_{x_{n-1}=0}^{x_{n-1}=x_n} \int_{x_{n-2}=0}^{x_{n-2}=x_{n-1}} \cdots \int_{x_2=0}^{x_2=x_3} \int_{x_1=0}^{x_1=x_2} dx$$

Claim: $\forall k \in \{2, \dots, n\}$, $\int_0^{x_k} \int_0^{x_{k-1}} \cdots \int_0^{x_2} dx_1 \cdots dx_{k-1} = \frac{(x_k)^{k-1}}{(k-1)!}$

Proof: By induction:

For $k=2$ we have:

$\int_0^{x_2} dx_1 = x_2$ ✓

Assume holds for some $k \in \{2, \dots, n\}$.

Check $k+1$:

5

$$\int_0^{x_{k+1}}\int_0^{x_k}\int_0^{x_{k-1}}\cdots\int_0^{x_2} dx_1\cdots dx_{k-1}dx_k =$$

$$= \int_0^{x_{k+1}} \frac{(x_k)^{k-1}}{(k-1)!}\,dx_k = \frac{(x_{k+1})^k}{k!} \quad ✓$$

Thus using the formula with $k=n$ we have:

$$\int_{D_n} dx = \int_0^1 \frac{(x_n)^{n-1}}{(n-1)!}\,dx_n = \frac{(x_n)^n}{n!}\Big|_0^1 = \boxed{\frac{1}{n!}}$$

(b) $$\int_{\Delta^n} \exp\Big(\sum_{j=1}^n x_j\Big)\,dx = \int_{D_n} \exp(x_n)\,dx_1\cdots dx_n$$

$$= \int_0^1\int_0^{x_n}\cdots\int_0^{x_2} \exp(x_n)\,dx_1\cdots dx_n$$

$$= \int_0^1 \exp(x_n)\,\frac{(x_n)^{n-1}}{(n-1)!}\,dx_n =: I_n$$

$$I_1 = e-1$$

$$I_n = \int_0^1 e^t\,\frac{t^{n-1}}{(n-1)!}\,dt = \int_0^1 (e^t)'\,\frac{t^{n-1}}{(n-1)!}\,dt$$

partial int.

$$= e^t\,\frac{t^{n-1}}{(n-1)!}\Big|_0^1 - \int_0^1 e^t\,\frac{t^{n-2}\,\cancel{(n-1)}}{\cancel{(n-1)}(n-2)!}\,dt$$

$$= \frac{e}{(n-1)!} - I_{n-1}$$

Claim: (Induction) $I_n = \left[\sum_{k=1}^{n-2}(-1)^{k-1}\,\frac{e}{(n-k)!}\right] + (-1)^n$

6

Q3 Let $\mathcal{D} := \left\{ \sum_{k=1}^{n} \underset{\in \mathbb{R}}{t_k} \underbrace{(a_k - a_0)}_{\in \mathbb{R}^n} \in \mathbb{R}^n \;\middle|\; (t_1, \dots, t_n) \in \Delta^n \right\}$

<u>Claim:</u> $vol_n(\mathcal{D}) = \frac{1}{n} \left| \det(a_1 - a_0, a_2 - a_0, \dots, a_n - a_0) \right|$

<u>Proof:</u> Define $\emptyset : \Delta^n \to \mathcal{D}$ by $\underset{\in \mathbb{R}^n}{t} \longmapsto \sum_{k=1}^{n} \underset{\in \mathbb{R}}{t_k} \underbrace{(a_k - a_0)}_{\in \mathbb{R}^n}$

$$\emptyset'(t) = \begin{bmatrix} \partial_1 \emptyset_1 & & \partial_n \emptyset_1 \\ \vdots & & \\ \partial_1 \emptyset_n & \cdots & \partial_n \emptyset_n \end{bmatrix} (t) = \left[\begin{pmatrix} a_1 - a_0 \end{pmatrix} ; \cdots ; \begin{pmatrix} a_n - a_0 \end{pmatrix} \right]$$

If $\emptyset'$ is singular, the vectors are not lin. indep. and then the vol. is zero.

Otherwise, $\emptyset$ is injective and we have:

$$\int_{\mathcal{D}} 1 = \int_{\emptyset(\Delta^n)} 1 = \int_{\Delta^n} (1 \circ \emptyset)(t) \left| \det(\emptyset'(t)) \right| dt$$

$$= \left| \det(a_1 - a_0, \dots, a_n - a_0) \right| \int_{\Delta^n} dt$$

Q4 (a) <u>Claim:</u> $f_a(x,y) = (x+y)^{-a} \quad \Delta^n \to \mathbb{R}$ is integrable iff $a < 2$ and then $\int_{\Delta^n} f = \frac{1}{2-a}$

<u>Proof:</u> Let $0 \leq s < t$.

Define $\Delta^2_{[s,t]} := \left\{ (x,y) \in [0, \infty)^2 \;\middle|\; s \leq x+y \leq t \right\}$

Define $J : [s,t] \times [0,1] \to \Delta^2_{[s,t]}$ by $(x,y) \longmapsto (x(1-y), xy)$

Then $J'(x,y) = \begin{bmatrix} 1-y & -x \\ y & x \end{bmatrix}$

$\Rightarrow \det(J'(x,y)) = (1-y)x + xy = x \geq 0$.

So this is injective (unless $x=0$, then measure $=0$).

So $\int_{\Delta^n} f = \int_{J([0,1]^2)} f = \int_{[0,1]^2} (f \circ J)(x) \left| J'(x) \right| dx =$

$$= \int_{[0,1]^2} \frac{1}{(x(1-y)+xy)^a} |x| \, dx \, dy$$

$$= \int_0^1 \int_0^1 x^{1-a} \, dx \, dy = \frac{1}{2-a} x^{2-a} \Big|_0^1 = \frac{1}{2-a}$$

$\boxed{\text{only exists when } a>2, \text{ then } 1-a<-1}$

(b) Claim: $f : \mathbb{R}_+^2 \setminus \Delta^2 \to \mathbb{R}$

$$(x,y) \longmapsto (x+y)^{-a}$$

is integrable iff $a > 2$, and then

$$\int_{\mathbb{R}_+^2 \setminus \Delta^2} f = \frac{1}{a-2}$$

Proof: Observe $\mathbb{R}_+^2 \setminus \Delta^2 = \Delta^2_{[1,\infty]}$ "

so that

$$\int_{\mathbb{R}_+^2 \setminus \Delta^2} f = \int_{\Delta^2_{[1,\infty]}} f = \int_{J([1,\infty]\times[0,1])} f =$$

$$= \int_{[1,\infty]\times[0,1]} f \circ J \, |\det(J')|$$

$$= \int_{[1,\infty]\times[0,1]} x^{1-a} \, dx \, dy = \int_1^\infty x^{1-a} \, dx$$

exists if $1-a<-1$ $\Rightarrow \boxed{a>2}$

$$= \boxed{\frac{1}{2-a}} \checkmark$$