Analysis II — HW #6

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Q1

$$S(x) = \int_0^1 (\dot{x}^2(t) - 1)^2 \, dt$$

$$L(x) = (\dot{x}^2(t) - 1)^2$$

Euler–Lagrange eq-n: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \underbrace{\frac{\partial L}{\partial x}}_{0} = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\Big(2(\dot{x}^2(t) - 1)\, 2\dot{x}(t)\Big)$$

$$= 4(\dot{x}^2(t) - 1)\ddot{x}(t) + 4(2\dot{x}(t)\ddot{x}(t))\dot{x}(t)$$

$$= 4\ddot{x}(t)\left[\dot{x}^2(t) - 1 + 2\dot{x}^2(t)\right]$$

$$= 4\ddot{x}(t)\left[3\dot{x}^2(t) - 1\right] \overset{!}{=} 0$$

$\boxed{\ddot{x}(t) = 0} \rightsquigarrow x(t) = At + B$

$x(0) = 0 \rightsquigarrow B = 0, \quad x(1) = 0 \rightsquigarrow A = 0$

$\boxed{3\dot{x}^2(t) - 1 = 0} \rightsquigarrow \dot{x}(t) = \pm\frac{1}{\sqrt{3}} \longrightarrow x(t) = \pm\frac{1}{\sqrt{3}}t + C$

$x(0) \overset{!}{=} 0 \rightsquigarrow C = 0$

$x(1) \overset{!}{=} 0 \rightsquigarrow \pm\frac{1}{\sqrt{3}} \overset{!}{=} 1 \rightsquigarrow \boxed{\perp}$

$\Rightarrow x(t) = 0$ is an extremum of S.

$S(0) = 1$

This extremum will NOT give you the infimum.
Think what non-differentiable function gives the actual minimum,
and then think how to make this smooth with a parameter that will eventually be sent to zero.

Q2

$$S(y) = \int_a^b \frac{\sqrt{1 + \dot{y}^2(x)}}{y(x)} \, dx \quad , y \in C^2([a,b], (0,\infty)), \; y(a) = y_0 \wedge y(b) = y_1$$

$L(y) = \frac{\sqrt{1+\dot{y}^2(x)}}{y(x)}$ Euler–Lagrange equations: $\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$

$\frac{\partial L}{\partial y} =$ $\qquad$ $\frac{\partial L}{\partial \dot{y}} =$

Next, observe that L does not depend explicitly on x.
As seen in the lecture, that means that the functional
$H = \dot{y}\frac{\partial L}{\partial \dot{y}} - L$ is a constant in x. Compute it:

$H =$

Euler-Lagrange eq-n is:

$$\boxed{\ddot{y}\, y^3 = -H^2}$$

$$\ddot{(y^2)} = \dot{(2y\dot{y})} = 2(\dot{y}^2 + y\ddot{y}) = \quad = -2$$

$$\Longrightarrow \quad \dot{(y^2)} = -2x + A_1$$

$$\Longrightarrow \quad y^2 = -x^2 + A_1 x + A_2 = -(x^2 - A_1 x) + A_2$$

$$= -\left(x^2 - 2\frac{A_1}{2}x + \left(\frac{A_1}{2}\right)^2 - \left(\frac{A_1}{2}\right)^2\right) + A_2$$

$$= -\left(x - \frac{A_1}{2}\right)^2 + \left(\frac{A_1}{2}\right)^2 + A_2$$

$$\Longrightarrow \quad y^2 + \left(x - \frac{A_1}{2}\right)^2 = \left(\frac{A_1}{2}\right)^2 + A_2$$

non-negative.

$\boxed{Q3}$ $f \in C^2(\mathbb{R}^n, \mathbb{R})$

A critical point of f is non-degenerate iff $\det(f''(x)) \neq 0$.

f' is a map $\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \cong \mathrm{Mat}_{n\times 1}(\mathbb{R})$

f'' is a map $\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R})) \cong \mathrm{Mat}_{n\times n}(\mathbb{R})$

$$f'(x) = \left[(\partial_1 f)(x) \;\; (\partial_2 f)(x) \;\; \dots \;\; (\partial_n f)(x)\right]$$

$$f''(x) = \begin{bmatrix} (\partial_1\partial_1 f)(x) & \cdots & (\partial_n\partial_1 f)(x) \\ \vdots & \ddots & \\ (\partial_1\partial_n f)(x) & \cdots & (\partial_n\partial_n f)(x) \end{bmatrix}$$

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n \qquad x \xmapsto{\nabla f} \begin{bmatrix} (\partial_1 f)(x) \\ \vdots \\ (\partial_n f)(x) \end{bmatrix}$$

Claim: All non-degenerate critical points are isolated. $\boxed{3}$

Proof: Let $x \in \mathbb{R}^n$ be non-degenerate and critical.

That is, $f'(x) = 0$ and $\det(f''(x)) \neq 0$.

We need to show $\exists\, r > 0$ s.t. $\forall\, y \in B_r(x)$, $f'(y) \neq 0$.

Note $f'(x) = 0 \iff (\nabla f)(x) = 0$.

Note $f''(x) = (\nabla f)'(x)$. $\implies \det(f''(x)) \neq 0 \iff \det((\nabla f)'(x)) \neq 0$

$\Updownarrow$

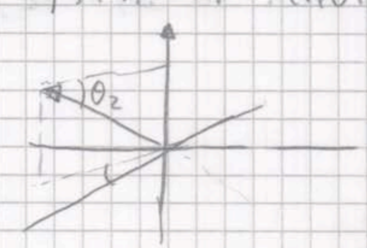
$(\nabla f)'(x)$ invertible

Use the inverse function theorem now on ∇f at x to conclude that

$\boxed{\text{Q5}}$ $f_n : \mathbb{R}^n \to \mathbb{R}^n \quad (r, \theta_1, \dots, \theta_{n-1}) \longmapsto r \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) \cdots \cos(\theta_{n-1}) \\ \sin(\theta_1)\cos(\theta_2) \cdots \cos(\theta_{n-1}) \\ \\ \sin(\theta_{n-1}) \end{bmatrix}$

Example: $f_2 : \mathbb{R}^2 \to \mathbb{R}^2 \quad (r, \theta_1) \mapsto r \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}$ polar coordinates.

$f_3 : \mathbb{R}^3 \to \mathbb{R}^3 \quad (r, \theta_1, \theta_2) \longmapsto r \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_2) \end{bmatrix}$ spherical coordinates



For (a), use induction.

For (b), use orthogonality via $\langle \partial_r f_n, \partial_{\theta_i} f_n \rangle = 0$ or

$\langle \partial_{\theta_i} f_n, \partial_{\theta_j} f_n \rangle = 0$.

For (c): Diffeomorphism $\equiv$ Bijective / C^∞ (inverse also) homeomorphism (cont. & inverse cont.)

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Because the partial derivatives are orthogonal, the Jacobian matrix is invertible. $\Longrightarrow$ Using the inverse function theorem, f_n is injective on U_n. Need to show surjectivity on the whole of V_n.

Q4

$\mathcal{A}$ is a Banach algebra.

$\psi \in C^1(\mathcal{A}, \mathcal{A})$

$x_0 \in \mathcal{A}$, $r > 0$, $\alpha \in (0,1)$ s.t. $\|\psi'(x) - \mathbb{1}\|_{\mathcal{L}(\mathcal{A},\mathcal{A})} \leq \alpha$

$\forall x \in B_r(x_0)$

Then: ① $\psi|_{B_r(x_0)}$ is injective.

② $\psi(B_r(x_0)) \in \text{Open}(\mathcal{A})$

③ $B_{r(1-\alpha)}(\psi(x_0)) \subseteq \psi(B_r(x_0)) \subseteq B_{r(1+\alpha)}(\psi(x_0))$

④ $\left[\psi^{-1} : \psi(B_r(x_0)) \longrightarrow B_r(x_0)\right] \in C^1(\mathcal{A}, \mathcal{A})$

Claim: $\forall\, y \in \mathcal{A} : \|y - \mathbb{1}\| < \frac{1}{2}\ \exists!\, x \in \mathcal{A} : \|x - \mathbb{1}\| < \frac{1}{2}$ s.t. $y^2 = x$.

That is, $\forall$ map $f : \mathcal{A} \to \mathcal{A}$ $\quad y \longmapsto \sqrt{y}$

Subclaim: $f \in C^1$

Proof: Define $\psi : \mathcal{A} \to \mathcal{A}$ $\quad x \longmapsto \frac{1}{2} x^2$

$\psi'(x) = \frac{1}{2}\{x, -\}$ (verify)

Use sentence above with $x_0 = \mathbb{1}$, $r = \frac{1}{2}$, $\alpha = 1/2$