Analysis II — HW # 8 Hints — 16/4/2015

Q2 $f : \mathbb{R} \to \mathbb{R}^n$ is some unknown function except $f(0) = f_0$.

$A : \mathbb{R}^n \to \mathbb{R}^n$ is some (constant) matrix.

How to solve $\dot{f}(t) = A(f(t))$? (Cartan diff calc pp. 125)

Solution exists and is given by $f(t) = (\exp(At))(f_0)$ Because $f(0) = f_0$ indeed and $\dot{f}(t) = A(f(t))$ as you can verify using HW4Q3.

Furthermore, assume A is diagonalizable with $A = PDP^{-1}$.

Then by HW2Q3 we have:

$$\exp(tA) = \exp(tPDP^{-1}) = P\exp(tD)P^{-1}$$

so that $\boxed{f(t) = P\exp(tD)P^{-1}f_0} = P\,\mathrm{diag}(e^{tD_{11}}, e^{tD_{22}})P^{-1}f_0$

⊛ Example: $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

$$\Rightarrow f(t) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} (f_0)_1 \\ (f_0)_2 \end{bmatrix} = \begin{bmatrix} (f_0)_1 e^{t} \\ \frac{1}{2}(f_0)_1 e^{t} - \frac{1}{2}e^{3t}((f_0)_1 - 2(f_0)_2) \end{bmatrix}$$

Q3 Same as above. Assume $A = -A^T$ is a real matrix.

W.T.S. that solutions of $\dot{f}(t) = A(f(t))$ live on a sphere.

$\Leftrightarrow \|f(t)\| = \text{const} \quad \forall t \in \mathbb{R} \Leftrightarrow \langle f(t), f(t)\rangle = \text{const} \quad \forall t \in \mathbb{R}$

$\Leftrightarrow \frac{\partial}{\partial t}\langle f(t), f(t)\rangle = 0 \quad \forall t \in \mathbb{R}$.

Use $\frac{\partial}{\partial t}\langle \alpha(t), \beta(t)\rangle = \langle \frac{\partial}{\partial t}\alpha(t), \beta(t)\rangle + \langle \alpha(t), \frac{\partial}{\partial t}\beta(t)\rangle$

Q4 $(A, B) \in \mathrm{Mat}_n(\mathbb{R})$ s.t. $[A, B] = 0$ $\quad ([A,B] \equiv AB - BA)$

W.T.S. that $\exp(A+B) = \exp(A)\exp(B)$

Main idea: Cartan pp. 112: $\dot{f}(t) = Af(t) + B$ has a <u>unique</u> solution by Theorem 1.9.1.

Define $f_1 : \mathbb{R} \to \mathrm{Mat}_n(\mathbb{R}) \quad t \mapsto \exp(tA)\exp(tB)$

$f_2 : \mathbb{R} \to \mathrm{Mat}_n(\mathbb{R}) \quad t \mapsto \exp(t(A+B))$

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We will show that actually both f_1 and f_2 fulfill

$$\circledast \begin{cases} \dot{f}_i(t) = (A+B) f_i(t) \\ f_i(0) = \mathbb{1} \end{cases}$$

As a result, $f_1(t) = f_2(t) \quad \forall\, t \in \mathbb{R}$. In particular for $t=1$ and so the claim follows.

To show $\circledast$ for f_2 is easy.

For f_1, use the Leibnitz rule and you will need to show

$$[\exp(tA), B] = 0$$

Q5

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \overset{\circledast}{=} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$ is a differential eq. of $\begin{bmatrix} x \\ y \end{bmatrix} : \mathbb{R} \to \mathbb{R}^2$

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \longmapsto A \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ (this is a v. field)

What is the flow of f? "It is a curve $\mathbb{R} \to \mathbb{R}^2$ which starts at the 2^{nd} slot and has derivative f."

It is a map $\varnothing : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) \longmapsto \varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix})$$

s.t.: $$\left(\begin{array}{l} ① \quad \varnothing(0, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ ② \quad \frac{d}{dt} \varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) = f(\varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix})) \quad \forall\, t \in \mathbb{R} \end{array} \right) \forall \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^2$$

Interpret ②: $f(\varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix})) = A \varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix})$

$\Rightarrow$ ② is equivalent to $\frac{d}{dt} \varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) = A \varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix})$

If we define $\varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ where $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution of $\circledast$ with initial value $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ then we will have found the flow of f.

$$\Longrightarrow \boxed{\varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) = P \operatorname{diag}(e^{tD_{11}}, e^{tD_{22}}) P^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ is the flow.}}$$

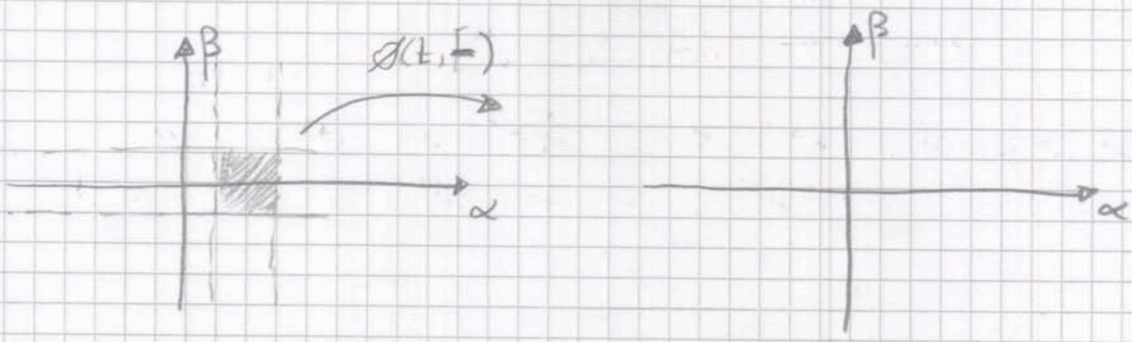
What to do if D_{11} or D_{22} are complex? $\varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix}) \overset{!}{\in} \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow D_{11} = 1+i,\ D_{22} = 1-i$$

$$A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

$\varnothing(t, \begin{bmatrix} \alpha \\ \beta \end{bmatrix})$ is real none the less after you work out the algebra. (Look up Christmas HW).

$$Q := \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; \alpha \in [1,3] \wedge \beta \in [-1,1] \right\}$$



What does $\phi(t,-)$ do geometrically?

Q1

Let $\mathcal{U}\in \mathrm{Open}(\mathbb{R}^n)$, $f:\mathcal{U}\to\mathbb{R}^n$ a continuously differentiable vector field on $\mathcal{U}$.

Define $\Omega := \{(t,x_0)\in\mathbb{R}\times\mathcal{U} \mid t\in I(x_0)\}$ where $I(x_0)$ is the "max. existence interval of x_0".

Define $\phi:\Omega\to\mathbb{R}^n$ as the flow of f.

That means that $\forall\ t\in I(x_0)$, $\begin{cases}\frac{d}{dt}\phi(t,x_0) = f(\phi(t,x_0)) & \text{for fixed } x_0\in\mathcal{U}\\ \phi(0,x_0)=x_0\end{cases}$

Assume ϕ is cont. differentiable.

Let $\xi_0\in\mathbb{R}^n$ be given, let $x_0\in\mathcal{U}$ be given.

Define $\xi: I(x_0)\to\mathbb{R}^n$ by $t\longmapsto(\phi'(t,x_0))(\xi_0)$

where ϕ' is the derivative w.r.t. the 2nd slot of ϕ alone.

Claim: $\begin{cases}\dot{\xi}(t) = (f'(\phi(t,x_0)))(\xi(t))\\ \xi(0)=\xi_0\end{cases}$

Proof: $\xi(0)\equiv(\phi'(0,x_0))(\xi_0)$

But $\phi(0,x_0)=x_0 \implies \phi(0,-)=\mathbb{1}(-)$

Thus, the Frechet derivative of $\mathbb{1}$ is $\mathbb{1}$.

$\implies \phi'(0,x_0)=\mathbb{1}$ ✓

$$\dot{\xi}(t)\equiv\frac{\partial}{\partial t}\left((\phi'(t,x_0))(\xi_0)\right)$$

$$=\frac{\partial}{\partial t}\left(\sum_{l=1}^n\sum_{j=1}^n(\partial_j\phi_l)(t,x_0)\,(\xi_0)_j\,\hat{e}_l\right)$$

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$\partial_j\partial_t\phi$ is cont.

$$=\sum_{l=1}^n\sum_{j=1}^n\left((\partial_j(\partial_t\phi_l))(t,x_0)\right)(\xi_0)_j\,\hat{e}_l$$

$$=\sum_{l,j=1}^n\left((\partial_j(f_l\circ\phi))(t,x_0)\right)(\xi_0)_j\,\hat{e}_l$$

$$=\sum_{l,j,m=1}^n\left(\left(((\partial_m f_l)\circ\phi)(\partial_j\phi_m)\right)(t,x_0)\right)(\xi_0)_j\,\hat{e}_l$$

$$=\left(f'(\phi(t,x_0))\right)\left(\phi'(t,x_0)(\xi_0)\right)$$

$$=\left(f'(\phi(t,x_0))\right)(\xi(t))$$

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