Analysis II — HW#9   22/4/2015   1

Q1 Let (X,d) be a metric space and let $A \subseteq X$ be given.

Recall that being a metric space, X has a natural topology associated with it, called the metric top.:

(1) $$\mathrm{Open}(X) \equiv \{ \mathcal{U} \subseteq X \mid \forall x \in \mathcal{U} \ \exists r > 0 \text{ s.t. } B_r(x) \subseteq \mathcal{U} \}$$

where $B_r(x) \equiv \{ y \in X \mid d(x,y) < r \}$ is the open ball around x with radius r.

Having the topology, such as the one in (1), a number of definitions are possible:
- The closure of a subset.
- The interior of a subset.
- The boundary of a subset.

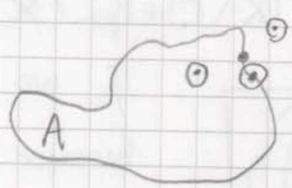
also subsets of X.

In the lecture you have defined:

— $\mathrm{Closure}(A) \equiv \bar{A} \equiv \{ x \in X \mid B_\varepsilon(x) \cap A \neq \emptyset \quad \forall \varepsilon > 0 \}$

"Abschluss"

Picture $A \subseteq \mathbb{R}^2$:



The boundary line may or may not be part of A, but it is certainly in $\bar{A}$, because if you pick any ball around any point on the boundary line, part of the ball will always lie in A. Note that $A \subseteq \bar{A}$ clearly.

This definition is not the most natural one for closure. The best def. to remember is (generalizes for non-metric topologies):

$$\boxed{\text{Closure of } A \equiv \text{"Smallest" closed set containing } A}$$

or in formula: $$\boxed{\mathrm{Closure}(A) \equiv \bigcap_{\substack{C \supseteq A \\ C \in \mathrm{Closed}(X)}} C}$$

Note: The word "smallest" here is meant in the sense of set inclusion. That means closure(A) is a subset of any closed set which contains A.

— $\mathrm{Int}(A) \equiv \{ x \in X \mid \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A \} \equiv A^\circ$

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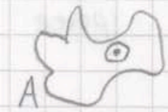
Again use the picture $A \subseteq \mathbb{R}^2$:



The points inside the boundary line comprise the interior of A, because we can always draw a circle around them which will lie entirely in A.

Again there is a more natural definition:

$\boxed{\text{Interior}(A) \equiv \text{The "largest" open subset of } A.}$

$$\boxed{\equiv \bigcup_{\substack{U \subseteq A \\ U \in \mathrm{Open}(X)}} U}$$

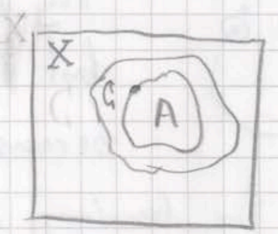
Note: "Largest" here means in the sense of set inclusion. That means $\mathrm{int}(A)$ contains any open subset of A.

Claim: Closure from lecture = General Closure def.

Proof: $\boxed{\subseteq}$ Let $x \in X$ be given s.t. $\forall\, \varepsilon > 0\ (B_\varepsilon(x) \cap A) \neq \emptyset$.

Let $C \in \mathrm{Closed}(X)$ s.t. $A \subseteq C$ be given.

Then $(X \setminus C) \in \mathrm{Open}(X)$ and $(X \setminus A) \supseteq (X \setminus C)$.



If we showed $x \in C$ we'll be finished because C was arbitrary and thus x must be in the intersection of all such sets of the form of C.

So we need to show $x \notin (X \setminus C)$. By $(X \setminus C) \in \mathrm{Open}(X)$ we have that if $x \in (X \setminus C)$, then $\exists\, \delta > 0$ s.t. $B_\delta(x) \subseteq (X \setminus C) \subseteq (X \setminus A)$.

But then we'd have $B_\delta(x) \cap A = \emptyset$, contradicting the hypothesis.

$\boxed{\supseteq}$ Let $x \in X$ be given s.t. $(\forall\, C \in \mathrm{Closed}(X)$ s.t. $C \supseteq A)$, $x \in C$.

Let $\varepsilon > 0$ be given. We need to show $(B_\varepsilon(x) \cap A) \neq \emptyset$.

Assume otherwise. Then $(X \setminus B_\varepsilon(x)) \in \mathrm{Closed}(X)$ and because $B_\varepsilon(x) \subseteq (X \setminus A)$, $(X \setminus B_\varepsilon(x)) \supseteq A$. But by def. of $B_\varepsilon(x)$, $x \notin X \setminus B_\varepsilon(x)$ which contradicts the hypothesis.

Claim: Interior from lecture = General interior def.

Proof: $\boxed{\subseteq}$ Let $x \in X$ be given s.t. ($\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A$).

Then $B_\varepsilon(x) \in Open(X)$ and $B_\varepsilon(x) \subseteq A$, so that $B_\varepsilon(x)$ is one of the elements in the union in the def. of $Int(A)$.

$\boxed{\supseteq}$ Let $x \in \bigcup_{\substack{U \in Open(A) \\ U \subseteq A}} U$ be given.

$\Rightarrow \exists\, U_0 \in Open(X)$ s.t. $x \in U_0 \subseteq A$.

By $U_0 \in Open(X)$, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U_0$.

As a result parts (b), (c), and (d) of $\boxed{Q1}$ are solved.

(a) and (e) are straight forward.

Just remember the general strategy to show two sets are equal: $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$

$$X \subseteq Y \Leftrightarrow \forall x \in X,\ x \in Y \qquad\qquad Y \subseteq X \Leftrightarrow \forall y \in Y,\ y \in X.$$

and apply it in both exercises.

In the lecture you have also defined the boundary of a set A: $\boxed{\partial A \equiv \bar{A} \setminus \mathring{A}}$

Example: ⊛ $A = (0,1) \subseteq \mathbb{R}$, closure$(A) = [0,1]$

interior$(A) = (0,1)$

$\partial A = \{0, 1\}$

⊛ closure$(\mathbb{R}) = \mathbb{R}$ (remember $\mathbb{R}$ is "clopen").

interior$(\mathbb{R}) = \mathbb{R}$

$\partial \mathbb{R} = \emptyset$

⊛ $A = \mathbb{Q} \subseteq \mathbb{R}$, closure$(\mathbb{Q}) = \mathbb{R}$

interior$(\mathbb{Q}) = \emptyset$ ($B_\varepsilon(q) \subseteq \mathbb{Q}$ is never true)

$\Rightarrow \partial \mathbb{Q} = \mathbb{R}$

Then $\partial(\partial \mathbb{Q}) = \emptyset$ so that $\boxed{\partial\partial A \neq \partial A}$ in general.

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⊛ $A = \{1\} \subseteq \mathbb{R} \Rightarrow \text{closure}(A) = \{1\}$ (remember singletons are closed in any sensible top. sp., of which $\mathbb{R}$ is one!)

$\text{int}(A) = \emptyset$ because $B_\varepsilon(1) \nsubseteq \{1\} \quad \forall \varepsilon > 0$.

$\Rightarrow \partial A = \{1\}$.

⊛ $A = S^n \equiv \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subseteq \mathbb{R}^{n+1}$

$S^n \in \text{Closed}(\mathbb{R}^{n+1})$ (show by looking at points in $(\mathbb{R}^{n+1} \setminus S^n)$).

$\Rightarrow \text{closure}(S^n) = S^n$

$\text{int}(S^n) = \emptyset$ because $\underset{\mathbb{R}^{n+1}}{\overset{\cap}{B_\varepsilon(x)}} \nsubseteq S^n \quad \forall x \in S^n$

$\Rightarrow \partial S^n = S^n$

⊛ $A = B^n \equiv \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ ($\equiv B_1(0)$) is the open unit ball around zero.

$B^n \in \text{Open}(\mathbb{R}^n) \Rightarrow \text{int}(B^n) = B^n$

$\text{closure}(B^n) = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ (this is HW, in $\boxed{Q3}$ (a)).

$\Rightarrow \partial B^n = \{x \in \mathbb{R}^n \mid \|x\| = 1\} = S^{n-1}$

⊛ $A = \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N} \setminus \{0\}\} \subseteq \mathbb{R}$

A is not closed: $\mathbb{R} \setminus A$ is not open because we cannot find a ball around $0 \in (\mathbb{R} \setminus A)$ which is inside of $\mathbb{R} \setminus A$.

A is not open: it has isolated points.

$\text{closure}(A) = A \cup \{0\}$ (smallest closed set containing A)

$\text{int}(A) = \emptyset$ because $B_\varepsilon(x) \subseteq A$ is never true.

$\Rightarrow \partial A = A \cup \{0\}$

⊛ $A = \emptyset \subseteq X$ (any top. sp.)

$\text{closure}(\emptyset) = \emptyset$ because $\emptyset \in \text{Closed}(X)$ (remember $\emptyset$ is also clopen)

$\text{int}(\emptyset) = \emptyset$ because $\emptyset \in \text{Open}(X)$

$\Rightarrow \partial \emptyset = \emptyset$.

$\boxed{Q2}$ (a) Claim: $\bar{A} \cup \bar{B} = \overline{A \cup B} \quad \forall \; (A,B) \in \mathcal{P}(X)^2$, X top. sp.

Proof: $\boxed{\supseteq}$ $(\bar{A} \cup \bar{B}) \in \text{Closed}(X)$ bcs. finite unions of closed sets are closed.

$(A \subseteq \bar{A}$ and $B \subseteq \bar{B}) \Rightarrow (A \cup B) \subseteq \bar{A} \cup \bar{B}$

$\Longrightarrow \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ by the def. of $\overline{A \cup B}$ as the smallest closed set containing $A \cup B$.

Note that if $\bar{A} \cup \bar{B}$ were not closed this would've failed (ºº)

$\boxed{\subseteq}$ Note that if $\mathcal{U} \subseteq \mathcal{V}$ then $\bar{\mathcal{U}} \subseteq \bar{\mathcal{V}}$ because $\bar{\mathcal{V}}$ is a closed set containing $\mathcal{U}$ ($\mathcal{U} \subseteq \mathcal{V} \subseteq \bar{\mathcal{V}}$) and $\bar{\mathcal{U}}$ must, by def., be inside of it.

$A \subseteq A \cup B \Rightarrow \bar{A} \subseteq \overline{A \cup B}$

$B \subseteq A \cup B \Rightarrow \bar{B} \subseteq \overline{A \cup B}$

$\Longrightarrow \bar{A} \cup \bar{B} \subseteq (\overline{A \cup B}) \cup (\overline{A \cup B}) = \overline{A \cup B}$

(b) According to $\boxed{Q1}$ (e), $X \setminus \mathring{A} = \overline{X \setminus A}$.

$\Longrightarrow \mathring{A} = X \setminus (\overline{X \setminus A})$.

$\Longrightarrow \mathring{A} \cap \mathring{B} =$

$=$

De Morgan

$\overset{(a)}{=}$

$=$

$= (A \cap B)^\circ$

(c) Claim: $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof: $\bar{A} \cap \bar{B}$ is closed (arbitrary int. of closed sets are closed)

If $x \in A \cap B$, then $x \in A \subseteq \bar{A}$ and $x \in B \subseteq \bar{B} \Rightarrow x \in \bar{A} \cap \bar{B}$.

$\Rightarrow A \cap B \subseteq \bar{A} \cap \bar{B}$

$\Rightarrow \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

(d) $\mathring{A} \cup \mathring{B} = [X \setminus (\overline{X \setminus A})] \cup [X \setminus (\overline{X \setminus B})] =$

$\subseteq$

6 For a counter example, try $A=(0,\frac{1}{2})$, $B=(\frac{1}{2},1)$, $X=\mathbb{R}$.

Necessary condition:
$[\mathring{A}\cup\mathring{B}=(A\cup B)^\circ] \Longleftrightarrow [\partial A\cap\partial B \subseteq \partial(A\cup B)]$

Proof HW

Q3 (a) Let X be a metric space. Let $r>0$ and $x\in X$ be given.

Claim: $\overline{B_r(x)} \subseteq \underbrace{\{y\in X \mid d(x,y)\leq r\}}_{C_r(x)}$

Proof: Claim: $B_r(x) \subseteq C_r(x)$

Proof:

Claim: $C_r(x)\in \mathrm{Closed}(X)$

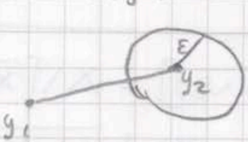
Proof:

Claim: If $\left(\forall\ (y_1,y_2)\in X^2 \text{ s.t. } y_1\neq y_2,\ \forall\ \varepsilon>0\right)\ \exists\ \underline{z\in X}$ s.t. $\left(d(z,y_1)<\varepsilon \text{ and } d(y_2,z)<d(y_1,y_2)\right)$ then $\overline{B_r(x)}=C_r(x)$.

Proof:

Claim: Let $(X,\|\cdot\|)$ be a normed vector space. Then the conditions of the previous claim hold.

Proof: Let $(y_1,y_2)\in X^2$ s.t. $y_1\neq y_2$ and $\varepsilon>0$ be given.



Define $z := \underbrace{\frac{1}{2}\frac{\varepsilon}{\|y_1-y_2\|}}_{t} y_1 + (1-\underbrace{\frac{1}{2}\frac{\varepsilon}{\|y_1-y_2\|}}_{t})y_2$.

(b) Counterexample: ⊛ $X = \mathbb{Z}$ with $d(n,m) := |n-m|$

This is indeed a metric space. (check!)

⊛ X is any set. $d(x,y) := \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

This is also a metric space. (check!)

$\boxed{Q4}$ Let C be the Cantor set.

Recall one way to write this down was (Rudin PMA $\boxed{2.44}$):

$$C = \bigcap_{n=1}^{\infty} E_n \quad \text{where} \quad E_1 = [0, 1/3] \cup [2/3, 1]$$

$$E_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$

$\ldots$

$\Longrightarrow \overline{C} = C$

Claim: $\mathring{C} = \emptyset$

Proof:

$\Rightarrow \partial C = C$

Q5 Recall "from the lecture" that if $(a,b) \in [\mathbb{R}^n]^2$ s.t. $a_j < b_j \;\; \forall \;\; j \in \{1, \dots, n\}$

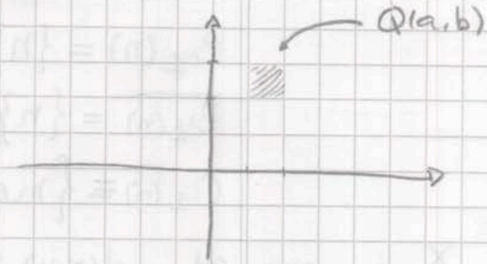
The closed n-cell, I^n, (denoted as $\overline{Q(a,b)}$ in lecture) is defined as

$$I^n := \left\{ x \in \mathbb{R}^n \;\middle|\; x_j \in [a_j, b_j] \quad \forall \; j \in \{1, \dots, n\} \right\}$$

Similarly the open n-cell, $\mathring{I}^n$, (denoted as $Q(a,b)$) is interior(I^n).

Example: $a = (1,2)$ $b = (2,3)$

$Q(a,b)$



Let $B \subseteq \mathbb{R}^n$ be a compact set.

(Recall by Heine-Borel that means <u>closed</u> and <u>bounded</u>).

A partition of B is a finite collection of ~~closed~~ n-cells $\{I_j^n\}_{j=1}^{\ell}$ for some $\ell \in \mathbb{N}$, s.t.: ① $B = \bigcup_{j=1}^{\ell} \overline{I_j^n}$

② $\mathring{I}_j^n \cap \mathring{I}_k^n = \emptyset \;\; \forall \;\; j \neq k$.

Define $Par(B) := \left\{ \{I_j^n\}_{j=1}^{\ell} \;\middle|\; \{I_j^n\}_{j=1}^{\ell} \text{ is a partition of } B \right\}$.

Not all sets in $\mathbb{R}^n$ admit a partition:

<u>Claim</u>: $B^2(0) \equiv \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$ does not admit a partition (but it is compact, being closed and bounded)

<u>Proof</u>:

$\partial B^2(0) = S^1$

Using the following claim(*), S^1 cannot be written as the union of fin. many sets of the form $\{a_1^j\} \times [a_2^j, b_2^j]$ and $[a_1^\ell, b_1^\ell] \times \{a_2^j\}$

Only very <u>few</u> of $\mathbb{R}^n$ cpt. subsets admit a partition!

The sets in $\mathbb{R}^n$ which admit a partition may seem like a very restricted collection, but don't worrys these will later contain the actually interesting sets,

<u>Claim</u>: $B \in \mathrm{Cpt}(\mathbb{R})$ admits a partition $\iff B = \bigcup_{j=1}^{\ell} [a_j, b_j]$ where

$a_j \leq b_j \ \forall j \in \{1, \dots, \ell\}$

and $b_j \leq a_{j+1} \ \forall j \in \{1, \dots, \ell-1\}$

<u>Proof</u>: (not much to prove)

⊛ <u>Claim</u>: $B \in \mathrm{Cpt}(\mathbb{R}^2)$ admits a partition $\iff \partial B$ is the union of finitely many axis-parallel intervals.

<u>Proof</u>: $\boxed{\Longrightarrow}$ $B = \bigcup_{j=1}^{\ell} [a_1^j, b_1^j] \times [a_2^j, b_2^j]$

Show $\partial B \subseteq \bigcup_{j=1}^{\ell} \Big[\left(\{a_1^j\} \times [a_2^j, b_2^j]\right) \cup \left(\{b_1^j\} \times [a_2^j, b_2^j]\right) \cup$

$\left([a_1^j, b_2^j] \times \{a_2^j\}\right) \cup \left([a_1^j, b_1^j] \times \{b_2^j\}\right) \Big]$

(try induction on ℓ)

$\boxed{\Longleftarrow}$ Assume $\partial B \subseteq \left(\bigcup_{i=1}^{\ell} \{a_1^i\} \times \mathbb{R} \right) \cup \left(\bigcup_{j=1}^{m} \mathbb{R} \times \{a_2^j\} \right)$

s.t. $a_1^i < a_1^{i+1} \ \forall i \in \{1, \dots, \ell-1\}$

$a_2^j < a_2^{j+1} \ \forall j \in \{1, \dots, m-1\}$

Define $Q_{ij} := (a_1^{i-1}, a_1^i) \times (a_2^{j-1}, a_2^j)$

<u>Claim</u>: $\{Q_{ij}\}$ are all disjoint

<u>Claim</u>: $\{\overline{Q_{ij}}\}$ covers B

<u>Claim</u>: $\forall (i,j)$, $(Q_{ij} \subseteq B) \veebar (Q_{ij} \cap B = \emptyset)$

Q6 Recall a Jordan-Null-Set is a set $A \subseteq \mathbb{R}^n$ s.t. $\forall \varepsilon > 0$ $\exists \underbrace{\{\mathring{I}_j\}_{j=1}^{\ell}}_{n\text{-cells}}$ s.t. $A \subseteq \bigcup_{j=1}^{\ell} \mathring{I}_j$ and $\sum_{j=1}^{\ell} \mathrm{vol}(I_j) < \varepsilon$

<u>Examples</u>:
- ⊛ $[a,b] \times \{0\}$ is a Jordan-Null-Set
- ⊛ ∂I^n is a Jordan-Null-Set
- ⊛ $[0,1]$ is not a Jordan-Null-Set: $\mathrm{vol}([0,1]) = 1$

10 (a) Claim: The Cantor set C is a Jordan-Null-Set.

Proof: $C \subseteq E_n \quad \forall\, n \in \mathbb{N}$ where E_n is a finite union of closed disjoint intervals, of total length $\left(\frac{2}{3}\right)^n$.

(b) Let $\{x_k\}_{k\in\mathbb{N}}$ be a convergent seq. in $\mathbb{R}^n$.

Claim: The image set $\{x_k \in \mathbb{R}^n \mid k \in \mathbb{N}\}$ is a Jordan-Null-Set.

Proof: $x := \lim_{j\to\infty} x_j$

Let $\varepsilon > 0$ be given.

Choose I_0 as an n-cell in $\mathbb{R}^n$ containing x s.t. $\mathrm{vol}(I_0) < \frac{1}{2}\varepsilon$.

We know $\exists\, N \in \mathbb{N}$ s.t. $\forall\, j > N$, $x_j \in \mathring{I}_0$.

Next, pick I_j as an n-cell in $\mathbb{R}^n$ containing x_j s.t. $\mathrm{vol}(I_j) < \frac{\varepsilon}{2N} \quad \forall\, j \in \{1, \dots, N\}$.

(c) Claim: $A := \mathbb{Q} \cap [0,1]$ is not a Jordan-Null-Set.

Proof: Let $\{I_j\}_{j=1}^{\ell}$ be any collection of open intervals which cover A. $I_j = (a_j, b_j)$, $A \subseteq \bigcup_{j=1}^{\ell} (a_j, b_j)$

WLOG assume $a_j < a_{j+1} \quad \forall\, j \in \{1, \dots, \ell-1\}$.

Claim: $a_{j+1} \leq b_j \quad \forall\, j \in \{1, \dots, \ell - 1\}$

Proof:

$\Rightarrow$ The cover must contain $[0, 1]$ fully.

$\Rightarrow \sum \mathrm{vol}(I_j) \geq 1 > \varepsilon \quad \forall\, \varepsilon > 0$.

Something is weird because A is countable.