Analysis II — Homework #3 — Solutions — 10/3/2015

Q1. <u>Claim</u>: The following functions $f:\mathbb{R}^n\to\mathbb{R}^m$ are everywhere differentiable.

The strategy would be: ① Either use Rudin PMA $\boxed{T}$ 9.17 to tentatively compute $f'(x)$ as $\text{Mat}_{m\times n}(\mathbb{R})\ni f'(x)=(\partial_i(f\cdot e_j))(x)$ and then verify that by plugging this into Rudin PMA $\boxed{D}$ 9.11 in retrospect. ② If we are able to verify that actually $\partial_i(f\cdot e_j)$ are all continuous, then by $\boxed{T}$ 9.21, f is continuously differentiable and thus differentiable.

We shall use both ① and ② alternatingly in what follows.

(a) $f:\mathbb{R}^2\to\mathbb{R}^2$ given by $\begin{bmatrix}x_1\\x_2\end{bmatrix}\longmapsto\begin{bmatrix}x_1^2-x_2^2\\2x_1x_2\end{bmatrix}$

<u>Proof</u>:

$$f'(x)\overset{?}{=}\begin{bmatrix}\partial_1(f\cdot e_1) & \partial_2(f\cdot e_1)\\ \partial_1(f\cdot e_2) & \partial_2(f\cdot e_2)\end{bmatrix}(x)=$$

from now on abbreviate $f\cdot e_j\equiv f_j$

$$=\begin{bmatrix}2x_1 & -2x_2\\ 2x_2 & 2x_1\end{bmatrix}$$

Now follow ①:

$$0\overset{?}{=}\lim_{h\to0}\frac{\|f(x+h)-f(x)-f'(x)h\|_{\mathbb{R}^2}}{\|h\|_{\mathbb{R}^2}}=$$

$$=\lim_{h\to0}\frac{\left\|\begin{bmatrix}(x_1+h_1)^2-(x_2+h_2)^2\\2(x_1+h_1)(x_2+h_2)\end{bmatrix}-\begin{bmatrix}x_1^2-x_2^2\\2x_1x_2\end{bmatrix}-\begin{bmatrix}2x_1 & -2x_2\\2x_2 & 2x_1\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix}\right\|_{\mathbb{R}^2}}{\|h\|_{\mathbb{R}^2}}=$$

$$=\lim_{h\to0}\frac{\left\|\begin{bmatrix}2x_1h_1+h_1^2-2x_2h_2-h_2^2\\2h_1x_2+2x_1h_2\end{bmatrix}-\begin{bmatrix}2x_1h_1-2x_2h_2\\2x_2h_1+2x_1h_2\end{bmatrix}\right\|_{\mathbb{R}^2}}{\|h\|_{\mathbb{R}^2}}=$$

$$=\lim_{h\to0}\frac{\left\|\begin{bmatrix}h_1^2+h_2^2\\0\end{bmatrix}\right\|_{\mathbb{R}^2}}{\left\|\begin{bmatrix}h_1\\h_2\end{bmatrix}\right\|_{\mathbb{R}^2}}=\lim_{h\to0}\frac{\sqrt{(h_1^2+h_2^2)^2}}{\sqrt{h_1^2+h_2^2}}=$$

$$=\lim_{h\to0}\sqrt{h_1^2+h_2^2}=\lim_{h\to0}\|h\|_{\mathbb{R}^2}=0\quad\checkmark$$

(b) $f:\mathbb{R}^3\to\mathbb{R}^2$ given by $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\longmapsto\begin{bmatrix}\sin(x_1x_2x_3)\\x_3^2\cos(x_1x_2^2)\end{bmatrix}$

<u>Proof</u>: $f'(x)\overset{?}{=}\begin{bmatrix}\partial_1f_1 & \partial_2f_1 & \partial_3f_1\\ \partial_1f_2 & \partial_2f_2 & \partial_3f_3\end{bmatrix}(x)=$

$$= \begin{bmatrix} \cos(x_1x_2x_3)\,x_2x_3 & \cos(x_1x_2x_3)\,x_1x_3 & \cos(x_1x_2x_3)\,x_1x_2 \\ -x_3^2\sin(x_1x_2^2)\,x_2^2 & -x_3^2\sin(x_1x_2^2)\,x_1 2x_2 & 2x_3\cos(x_1x_2^2) \end{bmatrix}$$

Now we shall use (2):

Each $\partial_i f_j : \mathbb{R}^3 \to \mathbb{R}$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto p(x_1,x_2,x_3)\, T(q(x_1,x_2,x_3))$$

where $T: \mathbb{R} \to \mathbb{R}$ is $\alpha \mapsto \cos(\alpha)$ or $\alpha \mapsto \sin(\alpha)$

and $p: \mathbb{R}^3 \to \mathbb{R}$, $q: \mathbb{R}^3 \to \mathbb{R}$ are polynomials of the form $p(x_1,x_2,x_3) = \sum_{l,m,n=1}^{M} \alpha_{l,m,n}\, x_1^l x_2^m x_3^n$

where $M \in \mathbb{N}$, $\alpha_{l,m,n} \in \mathbb{R}$ $\forall$ $(l,m,n) \in \{1,\dots,M\}^3$.

Thus we have
$$\partial_i f_j = p \cdot (T \circ q) = \mathcal{M} \circ (p, T \circ q)$$

multiplication in Banach algebra (in this case the Banach algebra is $\mathbb{R}$ and multiplication is ordinary multiplication of real numbers).

We have seen that $\mathcal{M}$ is continuous.

It is left as an exercise to show that p and q are cont.

Using Rudin $\boxed{T}$ 4.7 we know $T \circ q$ is continuous, $\mathcal{M} \circ (p, T \circ q)$

$\Rightarrow$ $\partial_i f_j$ are all continuous.

(c) $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3$

Proof:
$$f'(x) \overset{?}{=} \begin{bmatrix} 2x_1 - 2x_2x_3 & 2x_2 - 2x_1x_3 & 2x_3 - 2x_1x_2 \end{bmatrix}$$

Use (2).

(d) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto (x_2 - x_1^2)(x_2 - 2x_2^2)$

... and so on and so forth...

Q2 Claim: $f: \mathbb{R}^2 \to \mathbb{R}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{cases} 0 & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \dfrac{x_1^3}{\sqrt{x_1^2+x_2^2}} & \text{otherwise} \end{cases}$

is everywhere differentiable.

Proof: If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then we have:

$$f'(x) \overset{?}{=} \left[\frac{3x_1^2\sqrt{x_1^2+x_2^2} - x_1^3 \frac{\cancel{1/2}}{\sqrt{x_1^2+x_2^2}}\cancel{2}x_1}{\left(\sqrt{x_1^2+x_2^2}\right)^2} \quad x_1^3\left(-\cancel{\tfrac{1}{2}}\right)\frac{\cancel{2}x_2}{(x_1^2+x_2^2)^{3/2}} \right]$$

$$= \frac{x_1^2}{(x_1^2+x_2^2)^{3/2}}\left[2x_1^2+3x_2^2 \quad -x_1x_2 \right]$$

② As each partial derivative is continuous away from the origin, f is thus differentiable away from the origin.

At the origin we must be more careful:

$$(\partial_1 f)\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \lim_{t\to 0} \frac{f\left(\begin{bmatrix} t \\ 0 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{t} =$$

$$= \lim_{t\to 0} \frac{\frac{t^3}{\sqrt{t^2}}}{t} = \lim_{t\to 0} t = 0$$

$$(\partial_2 f)\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \lim_{t\to 0} \frac{f\left(\begin{bmatrix} 0 \\ t \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{t} =$$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

Thus our guess is $f'\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \overset{?}{=} [0 \ 0]$. Verify this: ①

$$0 \overset{?}{=} \lim_{h\to 0} \frac{\left\| f\left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \right\|_{\mathbb{R}}}{\|h\|_{\mathbb{R}^2}} =$$

$$= \lim_{h\to 0} \frac{\left| \frac{h_1^3}{\sqrt{h_1^2+h_2^2}} \right|}{\sqrt{h_1^2+h_2^2}} = \lim_{h\to 0} \frac{|h_1^3|}{(h_1^2+h_2^2)} = 0 \quad \checkmark$$

Q3 Let Y_1, Y_2 and Z be Banach sp.

Let $\beta: Y_1 \times Y_2 \to Z$ be a bilinear and bounded map.

That is, β is linear in each slot separately and $\exists\, C > 0$ s.t.:

$$\|\beta(y_1, y_2)\|_Z \leq C \, \|y_1\|_{Y_1} \|y_2\|_{Y_2} \quad \forall (y_1, y_2) \in Y_1 \times Y_2$$

[4] (a) <u>Claim</u>: β is differentiable and $\beta'(y_1,y_2) = [\beta(-,y_2) \quad \beta(y_1,-)]$.

<u>Proof</u>: As noted on the page, the product space $Y_1 \times Y_2$ has the norm: $\|(y_1,y_2)\|_{Y_1\times Y_2} = \|y_1\|_{Y_1} + \|y_2\|_{Y_2}$.

Thus we must show that

$$0 \stackrel{?}{=} \lim_{h\to 0} \frac{\| \beta((y_1,y_2)+(h_1,h_2)) - \beta((y_1,y_2)) - \beta'(y_1,y_2)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \|_Z}{\|h\|_{Y_1\times Y_2}} =$$

$$= \lim_{h\to 0} \frac{\| \beta(y_1+h_1, y_2+h_2) - \beta(y_1,y_2) - \beta(h_1,y_2) - \beta(y_1,h_2)\|_Z}{\|h\|_{Y_1\times Y_2}}$$

$$\stackrel{\beta \text{ is bi-linear}}{=} \lim_{h\to 0} \frac{\|\beta(h_1,h_2)\|_Z}{\|h\|_{Y_1\times Y_2}}$$

But β is bounded, so that

$$\|\beta(h_1,h_2)\|_Z \leq C \|h_1\|_{Y_1} \|h_2\|_{Y_2}$$

In addition, $\|h_i\|_{Y_i} \leq \|h\|_{Y_1\times Y_2} \quad \forall i \in \{1,2\}$, so:

$$\|\beta(h_1,h_2)\|_Z \leq C \left(\|h\|_{Y_1\times Y_2}\right)^2$$

$$\Rightarrow \quad 0 \leq \frac{\|\beta(h_1,h_2)\|_Z}{\|h\|_{Y_1\times Y_2}} \leq C \|h\|_{Y_1\times Y_2} \qquad \forall h \in Y_1 \times Y_2$$

$$\stackrel{h\to 0}{\Longrightarrow} \quad \lim_{h\to 0} \frac{\|\beta(h_1,h_2)\|_Z}{\|h\|_{Y_1\times Y_2}} = 0 \quad \text{as desired.}$$

Now, <u>clearly</u> $h \longmapsto \beta(h_1,y_2) + \beta(y_1,h_2)$ is a linear map: $ah + b\tilde{h} \longmapsto \beta(ah_1 + b\tilde{h}_1, y_2) + \beta(y_1, ah_2 + b\tilde{h}_2)$

$$= a\beta(h_1,y_2) + b\beta(\tilde{h}_1,y_2) + a\beta(y_1,h_2) + b\beta(y_1,\tilde{h}_2)$$

$$= a(\beta(h_1,y_2) + \beta(y_1,h_2)) + b(\beta(\tilde{h}_1,y_2) + \beta(y_1,\tilde{h}_2))$$

due to the bilinearity of β and in addition, the map is continuous:

Let $\varepsilon > 0$ be given. Then if $\|h - \tilde{h}\|_{Y_1\times Y_2} < \delta(\varepsilon)$ then

$$\|\beta(h_1,y_2) + \beta(y_1,h_2) - \beta(\tilde{h}_1,y_2) - \beta(y_1,\tilde{h}_2)\| =$$

multi-linearty

$$= \|\beta(h_1-\tilde{h}_1, y_2) + \beta(y_1, h_2-\tilde{h}_2)\| \leq \|\beta(h_1-\tilde{h}_1,y_2)\| + \|\beta(y_1,h_2-\tilde{h}_2)\|$$

boundedness

$$\leq C\|h_1-\tilde{h}_1\|_{Y_1}\|y_2\|_{Y_2} + C\|y_1\|_{Y_1}\|h_2-\tilde{h}_2\|_{Y_2} \leq$$

$$\leq C_1 \|h-\tilde{h}\|_{Y_1\times Y_2} \|y\|_{Y_1\times Y_2} + C_1 \|y\|_{Y_1\times Y_2} \|h-\tilde{h}\|_{Y_1\times Y_2}$$

$$\leq C_1 \|y\|_{Y_1\times Y_2}\, \delta(\varepsilon)$$

and so we can pick $\delta(\varepsilon) := \dfrac{\varepsilon}{C_1 \|y\|_{Y_1\times Y_2}}$ which works everywhere when $y \neq 0$ (what to do otherwise: exercise). ((hint: $\beta = 0$ if any of its slots is zero.))

and thus we see that β' is even uniformly continuous.
Thus β is indeed differentiable.

b) Let X_1, X_2 be two Banach sp. and $f_i \in Y_i^{X_i}$ $\forall i \in \{1,2\}$ be two differentiable maps.
Define $g: X_1 \times X_2 \longrightarrow Z$ by $g = \beta \circ (f_1, f_2)$

Claim: g is differentiable and
$$g'(x_1,x_2) = \left[\beta(f_1'(x_1)-, f_2(x_2)) \quad \beta(f_1(x_1), f_2'(x_2)-)\right]$$

Proof: Comparable to theorem 9.15 in Rudin's PMA, $\exists$ a theorem about Banach sp. replacing $\mathbb{R}^n$.
Define a map $f: X_1\times X_2 \to Y_1 \times Y_2$ by $(x_1,x_2) \longmapsto (f_1(x_1), f_2(x_2))$

Claim: f is differentiable with derivative
$$f'(x_1,x_2) = \begin{bmatrix} f_1'(x_1) & 0 \\ 0 & f_2'(x_2) \end{bmatrix}$$

Proof:
$$0 \overset{?}{=} \lim_{h\to 0} \frac{\|f(x_1+h_1, x_2+h_2) - f(x_1,x_2) - \begin{bmatrix} f_1'(x_1) & 0 \\ 0 & f_2'(x_2)\end{bmatrix}\begin{bmatrix} h_1 \\ h_2\end{bmatrix}\|_{Y_1\times Y_2}}{\|h\|_{X_1\times X_2}} =$$

$$= \lim_{h\to 0} \frac{\|f(x_1+h_1, x_2+h_2) - f(x_1,x_2) - \begin{bmatrix} f_1'(x_1)h_1 \\ f_2'(x_2)h_2\end{bmatrix}\|_{Y_1\times Y_2}}{\|h\|_{X_1\times X_2}} =$$

$$\frac{1}{\|h_i\|_{X_i}} \geq \frac{1}{\|h\|_{X_1\times X_2}} \ \forall i$$

$$= \lim_{h\to 0} \frac{\|f_1(x_1+h_1) - f_1(x_1) - f_1'(x_1)h_1\|_{Y_1} + \|f_2(x_2+h_2) - f_2(x_2) - f_2'(x_2)h_2\|_{Y_2}}{\|h_1\|_{X_1} + \|h_2\|_{X_2}}$$

$$\leq \lim_{h\to 0} \sum_{i=1}^{2} \frac{\|f_i(x_i+h_i) - f_i(x_i) - f_i'(x_i)h_i\|_{Y_i}}{\|h_i\|_{X_i}}$$

$= 0$ because f_i is differentiable by hypothesis.

But observe that $g = \beta \circ f$.
Now use the chain rule: $g'(x_1,x_2) = \beta'(f(x_1,x_2))\, f'(x_1,x_2)$
and using part (a) we get the desired result.

6 Q4

Let $\mathcal{U}\in \mathrm{Open}(\mathbb{R}^2)$ and let $f:\mathcal{U}\to\mathbb{R}$ be a continuously differentiable map s.t. $(\partial_2 f)=0$.

<u>Claim</u>: f depends only on x_1 if $\forall\, x_1\in\mathbb{R}$, the set defined by $\mathcal{U}_{x_1} := \{ x_2\in\mathbb{R} \mid (x_1,x_2)\in\mathcal{U}\}$ is <u>connected</u>.

<u>Proof</u>:

<u>Claim</u>: $\mathcal{U}_{x_1}\in \mathrm{Open}(\mathbb{R}) \quad \forall\, x_1\in\mathbb{R}$

<u>Proof</u>: Let $x_2\in\mathcal{U}_{x_1}$ be given.

Then $(x_1,x_2)\in\mathcal{U}$.

But $\mathcal{U}\in\mathrm{Open}(\mathbb{R}^2)$. $\implies \exists\, r>0$ s.t. $B_r(x_1,x_2)\subseteq\mathcal{U}$

$\implies \forall\, (y_1,y_2)\in\mathbb{R}^2$ s.t. $\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}<r$

$(y_1,y_2)\in\mathcal{U}$

But $(x_2-y_2)^2 \leq (x_1-y_1)^2+(x_2-y_2)^2$

$\implies |x_2-y_2| \leq r$

$\implies \exists\, r>0$ s.t. if $|x_2-y_2|<r$ then $y_2\in\mathcal{U}_{x_1}$.

<u>Claim</u>: The connected open subsets of $\mathbb{R}$ ~~are~~ ~~in~~ are intervals.

<u>Proof</u>: Munkres "Topology" Theorem 24.1.

$\implies \mathcal{U}_{x_1} = \underbrace{(a_{x_1}, b_{x_1})}_{\text{open interval}}$ for some $\underbrace{(a_{x_1}, b_{x_1})}_{\text{tuple}}\in\mathbb{R}^2$.

Define $I := \{x_1\in\mathbb{R} \mid \mathcal{U}_{x_1}\neq\emptyset\}$

Then $\mathcal{U} = \bigcup_{x_1\in I}\{x_1\}\times\mathcal{U}_{x_1} = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\in I, x_2\in(a_{x_1},b_{x_1})\}$

Let $x_1\in I$ be given.

Let $(x_2,\tilde{x}_2)\in[(a_{x_1},a_{x_1})]^2$ be given s.t. $x_2\neq\tilde{x}_2$.

We need to show that $f(x_1,x_2)-f(x_1,\tilde{x}_2)=0$

Define $\gamma:[0,1]\to\mathbb{R}^2$ by $t\mapsto\begin{bmatrix} x_1 \\ (1-t)x_2+t\tilde{x}_2\end{bmatrix}$

<u>Claim</u>: γ is continuous.

<u>Claim</u>: $\gamma([0,1])\subseteq\mathcal{U}$

<u>Proof</u>: Note that as $(x_2,\tilde{x}_2)\in[(a_{x_1},b_{x_1})]^2$ then any point in between is in the interval. In particular, $\forall\, t\in[0,1]$, $[(1-t)x_2+t\tilde{x}_2]\in(a_{x_1},b_{x_1})\equiv\mathcal{U}_{x_1}$

$\implies \gamma(t)\in\mathcal{U} \quad \forall\, t\in[0,1]$.

Also note that $f(x_1, x_2) = f \circ \gamma(0)$

$f(x_1, \tilde{x}_2) = f \circ \gamma(1)$

Thus $f(x_1, x_2) - f(x_1, \tilde{x}_2) =$

$= f \circ \gamma(0) - f \circ \gamma(1)$

$\boxed{T}$ 6.21 in Rudin. Show $(f \circ \gamma)'$ is integrable! Show $f \circ \gamma$ is differentiable

$$= \int_1^0 (f \circ \gamma)'(t)\, dt$$

$$= \int_1^0 \left[(f' \circ \gamma)(t)\right] \gamma'(t)\, dt$$

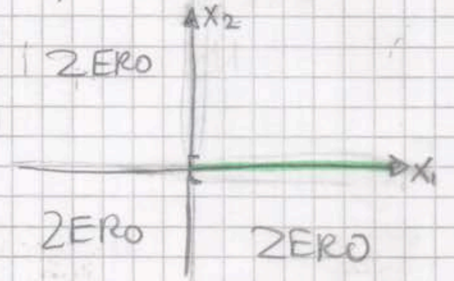
$$= \int_1^0 \left(\begin{bmatrix} \partial_1 f & \underbrace{\partial_2 f}_{0} \end{bmatrix} \circ \begin{bmatrix} x_1 \\ (1-t)x_2 + t\tilde{x}_2 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -x_2 + \tilde{x}_2 \end{bmatrix} dt =$$

$$= \int_1^0 \begin{bmatrix} \partial_1 f\left(\begin{bmatrix} x_1 \\ (1-t)x_2 + t\tilde{x}_2 \end{bmatrix}\right) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -x_2 + \tilde{x}_2 \end{bmatrix} dt =$$

$$= \int_1^0 0\, dt = 0$$

Claim: If $\mathcal{U}_{x_1}$ is not connected the preceding theorem fails.

Proof:



$\mathcal{U} := \mathbb{R}^2 \setminus [0, \infty) \times \{0\}$

open

$$f : \mathcal{U} \to \mathbb{R} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{cases} 0 & (x_1 < 0) \vee (x_2 \leq 0) \\ x_1^2 & (x_1 \geq 0) \wedge (x_2 > 0) \end{cases}$$

Then f is cont. diff. on its domain (check!),

$\partial_2 f = 0$, yet $\begin{cases} f(1, -1) = 0 \\ f(1, 1) = 1 \end{cases} \Rightarrow f$ is not independent of x_2.

$$\mathcal{U}_{x_1}\Big|_{x_1 = 1} = \left\{ x_2 \in \mathbb{R} \;\middle|\; (1, x_2) \in \mathcal{U} \right\} = (-\infty, 0) \cup (0, \infty)$$

not connected!