$\boxed{Q1}$ Let $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{cases} (x_1^2+x_2^2)\sin\left(\frac{1}{x_1^2+x_2^2}\right) & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$

(a) Claim: f is differentiable.

Proof: $f'\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \stackrel{?}{=} \begin{bmatrix} \partial_1 f & \partial_2 f \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\partial_i f =$

$=$

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, $\partial_i f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \lim$

$=$

Claim: $(\partial_i f)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ is cont. away from the origin.

Proof:

f is cont. diff. away from the origin.

Note that $\lim\limits_{h\to 0} \dfrac{\left\| f(h_1,h_2) - f(0) - [\partial_1 f(0) \;\; \partial_2 f(0)]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|}{\|h\|} =$

[2] (b) **Claim:** f is not cont. diff. at the origin.

Proof: We will show that the map

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto [(\partial_1 f)(x_1,x_2) \quad (\partial_2 f)(x_1,x_2)]$$

is not cont. at the origin.

Alternatively we can follow Rudin $\boxed{T}$ 9.21 to show that $\partial_i f : \mathbb{R}^2 \to \mathbb{R}$ is not cont. at the origin.

$$\lim_{x \to 0} \partial_1 f = \lim_{x_1 \to 0} (\partial_1 f)(x_1, 0) =$$

if limit exists then it cannot depend on the direction. So approach from

$=$

$\boxed{Q2}$ Let $\emptyset \in \mathbb{R}^{\mathbb{R}}$ be given s.t. $\emptyset$ is differentiable, $\emptyset(x) > 0 \ \forall x \in (1,2)$ and $\emptyset(x) = 0 \ \forall x \in \mathbb{R} \setminus (1,2)$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1,x_2) := \begin{cases} \sqrt{x_1^2 + x_2^2}\, \emptyset\left(\frac{x_2}{x_1^2}\right) & x_1 \neq 0 \\ 0 & x_1 = 0 \end{cases}$

Claim: f is cont.

Proof: **Claim:** f is diff. whenever $x_1 \neq 0$.

Proof: $f'(x) =$

$=$

whenever $x_1 \neq 0$.

The partial derivatives are cont. (argue why) and so f is (cont.) differentiable $\forall\, x_1 \neq 0$.

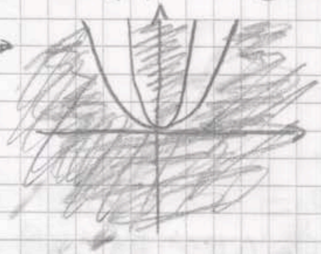
As diff. implies cont. (Rudin $\boxed{T}$ 9.13 (c)), we see that f is cont. $\forall\, x_1 \neq 0$.

Let $x_2 \in \mathbb{R}$ be given.

Then $\lim_{x_1 \to 0} f(x_1,x_2) = \lim_{x_1 \to 0} \sqrt{x_1^2 + x_2^2}\, \emptyset\left(\frac{x_2}{x_1^2}\right)$

Note that $\emptyset$ is zero on $\mathbb{R} \setminus (1,2)$. That means that $\emptyset(\frac{x_2}{x_1^2})$ is zero on the x_1–x_2 plane in the shaded region; which corresponds to $x_2 \geq 2x_1^2$ or $x_2 \leq x_1^2$.

shades include boundaries



Thus, by sequential continuity, f is cont. $\forall\, x \in \mathbb{R}^2$.

<u>Claim</u>: f is differentiable except at the origin.

<u>Proof</u>: Above we have seen that f is diff. $\forall\, x_1 \neq 0$.

So it remains to be shown that f is diff. on $\{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 \neq 0\}$.

$$(\partial_1 f)\left(\begin{bmatrix}0\\x_2\end{bmatrix}\right) = \lim_{t\to 0}$$

$$= \lim_{t\to 0}$$

$$(\partial_2 f)\left(\begin{bmatrix}0\\x_2\end{bmatrix}\right) = \lim_{t\to 0}$$

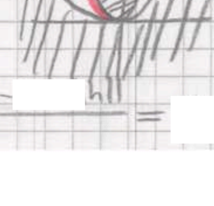
$$f'\left(\begin{bmatrix}0\\x_2\end{bmatrix}\right) \overset{?}{=} \qquad x_2 \neq 0$$

Verify: $(x_2 \neq 0)$

$$0 \overset{?}{=} \lim_{h\to 0} \frac{\left\| f\left(\begin{bmatrix}h_1\\x_2+h_2\end{bmatrix}\right) - f\left(\begin{bmatrix}0\\x_2\end{bmatrix}\right) - \quad h \right\|}{\|h\|} =$$

$$= \lim_{h\to 0}$$

for example:



Then $\lim_{h\to 0} \frac{\| f(x+h) - f(x) - \quad h \|}{\|h\|} = \quad \neq 0\,!$

[4] [Q3] Let $\mathcal{A}$ be a Banach algebra.
Let $\mathcal{A}^* := \{A \in \mathcal{A} \mid \exists A^{-1} \in \mathcal{A}\}$.

Claim: $\mathcal{A}^* \in \mathrm{Open}(\mathcal{A})$

Proof: Let $A \in \mathcal{A}^*$ be given.

Define $r := \|A^{-1}\|^{-1} > 0$.

Then if $B \in B_r(A)$, $\|B - A\| < \|A^{-1}\|^{-1}$

$\Longrightarrow \|A^{-1}\|\|B-A\| < 1$

$\Longrightarrow \|A^{-1}B - \mathbb{1}\| < 1$

Claim: If $\|X\| < 1$ then $(\mathbb{1} - X) \in \mathcal{A}^*$

Proof: Claim: $\{\sum_{j=0}^{n} X^j\}_{n\in\mathbb{N}}$ is Cauchy.

Proof: $\|\sum_{j=n_1}^{n_2} X^j\| \leq \sum_{j=n_1}^{n_2} \|X\|^j$

$$= \|X\|^{n_1}(1 + \|X\| + \ldots + \|X\|^{n_2 - n_1})$$

$$\underset{\substack{n_2-n_1 \text{ bounded}\\ n_1\to\infty}}{\leq} (n_2 - n_1)\|X\|^{n_1} \xrightarrow{} 0$$

But $\mathcal{A}$ is complete $\Longrightarrow \{\sum_{j=0}^{n} X^j\}_{n\in\mathbb{N}} \xrightarrow{n\to\infty} Y \in \mathcal{A}$.

Claim: $Y = (\mathbb{1} - X)^{-1}$

Proof: $Y(\mathbb{1} - X) = \left(\lim_{n\to\infty} \sum_{j=0}^{n} X^j\right)(\mathbb{1} - X) =$

mul. is cont.

$$= \lim_{n\to\infty}\left(\left(\sum_{j=0}^{n} X^j\right)(\mathbb{1} - X)\right) =$$

$$= \lim_{n\to\infty}\left[\left(\sum_{j=0}^{n} X^j\right) - \left(\sum_{j=0}^{n} X^{j+1}\right)\right] =$$

$$= \lim_{n\to\infty}\left[\left(\sum_{j=0}^{n} X^j\right) - \left(\sum_{j=1}^{n+1} X^j\right)\right] =$$

$$= \lim_{n\to\infty}\left(\mathbb{1} - X^{n+1}\right)$$

$$= \mathbb{1} - \lim_{n\to\infty} X^{n+1}$$

and similarly it can be verified that

$(\mathbb{1} - X)Y = \mathbb{1} - \lim_{n\to\infty} X^{n+1}$

Claim: $\lim_{n\to\infty} X^n = 0$

Proof: $\|\cdot\|$ is cont.

$\Longrightarrow \lim_{n\to\infty} \|X^n\| = \|\lim_{n\to\infty} X^n\|$

But $\|X^n\| \leq \|X\|^n \to 0$

$\Rightarrow \|\lim_{n\to\infty} X^n\| = 0$

$\Rightarrow \lim_{n\to\infty} X^n = 0$

$$\underbrace{\left[\mathbb{1} - (A^{-1}B - \mathbb{1})\right]}_{-A^{-1}B \in \mathcal{A}^*} \in \mathcal{A}^*$$

$\Rightarrow A^{-1}B \in \mathcal{A}^*$

$\Rightarrow B \in \mathcal{A}^*$

(a) Use the formula $(\exp'(A))(B) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{k=0}^{n-1} A^k B A^{n-k-1}$

(b) Claim: $(A \mapsto A^{-1})'(B) = -A^{-1}BA^{-1}$ (clearly linear & cont.)

Proof: Let $A \in \mathcal{A}^*$ be given. Then $\forall B \in \mathcal{A}^*$,

$$(A+B)^{-1} = (AA^{-1}A + AA^{-1}B)^{-1} = \left[A(\mathbb{1} + A^{-1}B)\right]^{-1}$$
$$= (\mathbb{1} + A^{-1}B)^{-1} A^{-1}$$
$$= (\mathbb{1} - (\mathbb{1} - A^{-1}B))^{-1} A^{-1}$$

$\exists\, \mathcal{U} \in \mathrm{Open}(\mathcal{A})$ s.t. $A \in \mathcal{U}$ and $\|\mathbb{1} - A^{-1}B\| < 1$ $\forall B \in \mathcal{U}$ (proof using cont. of $B \mapsto \mathbb{1} - A^{-1}B$).

Then for such B we may write

$(A+B)^{-1} =$

(Taylor's expansion of $\frac{1}{1-X}$ $\forall X \in \mathcal{A}^*$ s.t. $\|X\| < 1$. Why is this possible?)

Then

$$0 \overset{?}{=} \lim_{B\to 0} \frac{\|(A+B)^{-1} - A^{-1} - (-A^{-1}BA^{-1})\|}{\|B\|} =$$

$$= \lim_{B\to 0} \frac{\|\cancel{A^{-1}} - \cancel{A^{-1}BA^{-1}} + o(B^2) \cancel{-A^{-1}} + \cancel{A^{-1}BA^{-1}}\|}{\|B\|}$$

$= 0$ ✓

The series with powers of B: use monotone convergence to take limit inside series.

Q6 Let $f: \mathbb{R}^n \to \mathbb{R}$ be given.

Then $$f(x+a) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left(\left[\sum_{k=1}^{n} a_k \partial_k \right]^j f \right)(x) \right\}$$

$$= f(x) + \left(\left[\sum_{k=1}^{n} a_k \partial_k \right] f \right)(x) + \frac{1}{2} \left(\left[\sum_{k=1}^{n} a_k \partial_k \right]^2 f \right)(x) + \dots$$

$$= f(x) + \sum_{k=1}^{n} a_k (\partial_k f)(x) + \frac{1}{2} \left(\left[\sum_{k=1}^{n} a_k \partial_k \right]^2 f \right)(x) + \dots$$

For example: $\boxed{n=2}$

$$f(x+a) = f(x) + a_1(\partial_1 f)(x) + a_2(\partial_2 f)(x) + $$
$$+ \frac{1}{2}\left(\left(a_1^2 \partial_1^2 + 2a_1 a_2 \partial_1 \partial_2 + a_2^2 \partial_2^2 \right) f \right)(x) + \dots =$$

$$= f(x) + a_1 f_1(x) + a_2 f_2(x) +$$
$$+ \frac{1}{2} a_1^2 f_{11}(x) + \frac{1}{2} a_2^2 f_{22}(x) + a_1 a_2 f_{12}(x) + \dots$$

where $f_{ij} \equiv \partial_i \partial_j f$

$\boxed{n=3}$ $$f(x+a) = f(x) + \sum_{i=1}^{3} a_i f_i(x) + \frac{1}{2} \sum_{i=1}^{3} a_i^2 f_{ii}(x) +$$
$$a_1 a_2 f_{12}(x) + a_1 a_3 f_{13}(x) + a_2 a_3 f_{23}(x)$$

Q5 Use induction on (k) and the product rule.