Analysis II – Colloquium of Week I – 17/2/2015

Question 1

Def: Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a real v.sp. are called equivalent iff $\exists$ $c>0$ s.t. $\frac{1}{c}\|x\| \le \|x\|' \le c\|x\| \quad \forall x \in X$.

Claim: "Equivalence of norms" is an equivalence relation on the set of norms on X.

Proof: $\boxed{\text{Reflexivity}}$ Pick $c=1$. Then $\|x\| \le \|x\| \le \|x\| \quad \forall x \in X$.

$\boxed{\text{Symmetry}}$ If $\|\cdot\|$ is equiv. to $\|\cdot\|'$, then $\exists$ $c>0$ s.t.
$$\frac{1}{c}\|x\| \le \|x\|' \le c\|x\| \quad \forall x \in X.$$
$$\Longrightarrow \begin{cases} \|x\| \le c\|x\|' \le c^2\|x\| \\ \frac{1}{c}\|x\| \le \frac{1}{c}\|x\|' \le \|x\| \end{cases} \Longrightarrow \begin{cases} \|x\| \le c\|x\|' \\ \frac{1}{c}\|x\|' \le \|x\| \end{cases}$$

$\boxed{\text{Transitivity}}$ Let $\|\cdot\|$ be eq. to $\|\cdot\|'$ and $\|\cdot\|'$ be eq. to $\|\cdot\|''$.
Then $\exists$ $c>0$, $\tilde{c}>0$ s.t. $\begin{cases} \frac{1}{c}\|x\| \overset{①}{\le} \|x\|' \overset{②}{\le} c\|x\| \\ \frac{1}{\tilde{c}}\|x\|' \overset{③}{\le} \|x\|'' \overset{④}{\le} \tilde{c}\|x\|' \end{cases}$

$\overset{③}{\Longrightarrow} \|x\|' \le \tilde{c}\|x\|''$ | $\overset{④}{\Longrightarrow} \frac{1}{\tilde{c}}\|x\|'' \le \|x\|'$

$\overset{①}{\Longrightarrow} \frac{1}{c}\|x\| \le \tilde{c}\|x\|''$ | $\overset{②}{\Longrightarrow} \frac{1}{\tilde{c}}\|x\|'' \le c\|x\|$

$\Longrightarrow \frac{1}{c\tilde{c}}\|x\| \le \|x\|''$ | $\Longrightarrow \|x\|'' \le c\tilde{c}\|x\|$

But $c\tilde{c}>0$ as $c>0 \wedge \tilde{c}>0$. ∎

Q2

Let $\mathcal{U} \in [\mathrm{Open}(\mathbb{R}^n) \setminus \{\emptyset\}] \cap \mathrm{Convex}(\mathbb{R}^n) \cap \mathrm{Bounded}(\mathbb{R}^n)$ s.t.
$$[(x\in\mathcal{U}) \Longleftrightarrow (-x \in \mathcal{U})] \quad \forall x \in \mathbb{R}^n$$

Claim: $\|x\| := \inf(\{\lambda>0 \mid \frac{x}{\lambda} \in \mathcal{U}\}) \quad \forall x \in \mathbb{R}^n$ (Minkowski Functional) is a norm on $\mathbb{R}^n$.

Proof: **Claim:** $\|\mathbb{R}^n\| \subseteq \mathbb{R} \Longrightarrow \|\cdot\| \in \mathbb{R}^{\mathbb{R}^n}$

Proof: By def., $\|x\|$ is an inf. on a set of (positive) real numbers. The only possibility for this claim to fail then is if the set is empty, as $\inf(\emptyset) \equiv \infty$.
However, this is not possible:
$\mathcal{U} \ne \emptyset \Longrightarrow \exists\, y \in \mathcal{U} \Longrightarrow -y \in \mathcal{U}$
$\Longrightarrow ty + (1-t)(-y) \in \mathcal{U} \quad \forall t \in [0,1]$
Pick $t=1/2$. Then $0 \in \mathcal{U}$.
$\mathcal{U} \in \mathrm{Open}(\mathbb{R}^n) \Longrightarrow \exists$ $r>0$ s.t. $B_r(0) \subseteq \mathcal{U}$ where

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$B_r(0) \equiv \{x \in \mathbb{R}^n \mid \|x\|_2 < r\}$ and $\|x\|_2 \equiv \sqrt{\sum_{j=1}^{n} x_j^2}$

Note that $\forall R>0$, $RB_r(0) \equiv \{Rx \mid \|x\|_2 < r\}$
$= \{x \mid \|x/R\|_2 < r\}$
$= B_{Rr}(0)$

Next, $\forall x \in \mathbb{R}^n$, $\exists$ some $R>0$ s.t. $B_{Rr}(0) \ni x$.

Thus $\frac{x}{R} \in B_r(0) \subseteq \mathcal{U}$.

Thus $R \in \{\lambda > 0 \mid \frac{x}{\lambda} \in \mathcal{U}\}$.

$\Longrightarrow \{\lambda>0 \mid \frac{x}{\lambda} \in \mathcal{U}\} \neq \emptyset \quad \forall x \in \mathbb{R}^n$

$\Longrightarrow \|x\| < \infty \quad \forall x \in \mathbb{R}^n$.

<u>Claim</u>: $\|\alpha x\| = |\alpha| \|x\| \quad \forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n$.

<u>Proof</u>: If $\alpha = 0$ then $\alpha x = 0$.

$\|0\| = 0$ as $0 \in \mathcal{U}$. So this case works out.

If $\alpha > 0$, $\|\alpha x\| \equiv \inf(\{\lambda > 0 \mid \frac{\alpha x}{\lambda} \in \mathcal{U}\})$

$= \inf(\{\lambda>0 \mid \frac{x}{(\frac{\lambda}{\alpha})} \in \mathcal{U}\})$

$= \inf(\{\alpha\lambda > 0 \mid \frac{x}{\lambda} \in \mathcal{U}\})$

$= \inf(\alpha\{\lambda>0 \mid \frac{x}{\lambda} \in \mathcal{U}\})$

<u>Claim</u>: $\inf(\alpha A) = \alpha \inf(A) \quad \forall \alpha > 0, A \subseteq \mathbb{R}$.

<u>Proof</u>: Let $\varepsilon > 0$ be given. Then $\exists a_\varepsilon \in A$ s.t. $\inf(\alpha A) \le \alpha a_\varepsilon < \inf(\alpha A) + \varepsilon$

But $\alpha \inf(A) \le \alpha a_\varepsilon$.

$\overset{\varepsilon \to 0}{\Longrightarrow} \alpha \inf(A) \le \inf(\alpha A)$

Furthermore, $\exists \tilde{a}_\varepsilon \in A$ s.t.

$\inf(A) \le \tilde{a}_\varepsilon < \inf(A) + \varepsilon/\alpha$

$\Longrightarrow \quad \alpha \inf(A) \le \alpha \tilde{a}_\varepsilon \le \alpha \inf(A) + \varepsilon$

But $\inf(\alpha A) \le \alpha \tilde{a}_\varepsilon$.

$\overset{\varepsilon \to 0}{\Longrightarrow} \inf(\alpha A) \le \alpha \inf(A)$.

$= \alpha \inf(\{\lambda>0 \mid \frac{x}{\lambda} \in \mathcal{U}\}) \equiv |\alpha| \|x\|$.

If $\alpha < 0$, $\|\alpha x\| \equiv \inf(\{\lambda > 0 \mid \frac{\alpha x}{\lambda} \in \mathcal{U}\})$ $(x \in \mathcal{U}) \Longleftrightarrow (-x \in \mathcal{U})$

$= \inf(\{\lambda > 0 \mid -\frac{\alpha x}{\lambda} \in \mathcal{U}\})$

$= -\alpha \inf(\{\lambda>0 \mid \frac{x}{\lambda} \in \mathcal{U}\}) \equiv |\alpha| \|x\|$.

Claim: $\|x_1+x_2\| \leq \|x_1\|+\|x_2\| \quad \forall\ (x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n$

Proof: Let $\varepsilon>0$ be given.

Then $\exists\ \lambda_\varepsilon^i>0$ s.t. $\left(\frac{x_i}{\lambda_\varepsilon^i}\in\mathcal{U}\right)\wedge\left[\|x_i\|\leq\lambda_\varepsilon^i<\|x_i\|+\varepsilon\right]\ \forall i\in\{1,2\}$

$$\Rightarrow \|x_1\|+\|x_2\|\leq\lambda_\varepsilon^1+\lambda_\varepsilon^2<\|x_1\|+\|x_2\|+2\varepsilon$$

But
$$\frac{x_1+x_2}{\lambda_\varepsilon^1+\lambda_\varepsilon^2}=\underbrace{\frac{\lambda_1^\varepsilon}{\lambda_1^\varepsilon+\lambda_2^\varepsilon}}_{\substack{t\\ \in\\ [0,1]}}\underbrace{\frac{x_1}{\lambda_1^\varepsilon}}_{\in\ \mathcal{U}}+\underbrace{\frac{\lambda_2^\varepsilon}{\lambda_1^\varepsilon+\lambda_2^\varepsilon}}_{1-t}\underbrace{\frac{x_2}{\lambda_2^\varepsilon}}_{\in\ \mathcal{U}}\in\mathcal{U}$$

by convexity

$\Rightarrow \|x_1+x_2\|\leq\lambda_1^\varepsilon+\lambda_2^\varepsilon$ by def. of the infimum.

$\Rightarrow \|x_1+x_2\|<\|x_1\|+\|x_2\|+2\varepsilon\quad\forall\varepsilon>0$

$\overset{\varepsilon\to0}{\Rightarrow}$ The claim follows.

Claim: $\|x\|>0\quad\forall\ x\in\mathbb{R}^n\setminus\{0\}$.

Proof: Let $x\in\mathbb{R}^n\setminus\{0\}$ be given and assume $\|x\|=0$.

Then $\forall\ \varepsilon>0\ \exists\ \lambda_\varepsilon>0$ s.t. $0\leq\lambda_\varepsilon<\varepsilon$ and $\frac{x}{\lambda_\varepsilon}\in\mathcal{U}$.

$\Rightarrow \forall\ M>0\ \exists\ N(M)>0$ s.t. $M<N(M)$ and $N(M)x\in\mathcal{U}$.

But $\mathcal{U}$ is bounded. $\Rightarrow \boxed{\perp}$

∎

Q3

$C^0([0,1])\equiv\left\{f\in\mathbb{R}^{[0,1]}\ \middle|\ f \text{ is continuous}\right\}$ $\boxed{\|f\|_\infty:=\sup(\{|f(x)|\mid x\in[0,1]\})}$

Let $p\in[1,\infty)$ be given.

Define $\forall\ f\in C^0([0,1])$, $\boxed{\|f\|_p:=}\left\{\underbrace{\int_0^1|f(x)|^p\,dx}_{\text{Riemann integral (for now}\to\text{later will be Lebesgue)}}\right\}^{\frac{1}{p}}$

This definition is valid because a continuous function from a compact space has a compact range $\Rightarrow f([0,1])$ is compact.

$\overset{H.B.}{\Rightarrow} f$ is bounded.

$\Rightarrow f$ is Riemann integrable by definition.

$\Rightarrow |f|^p$ is Riemann integrable.

→ **Claim:** $C^0([0,1])$ is a v.sp. over $\mathbb{R}$. (easy)

→ **Claim:** $\|\cdot\|_p$ is a norm on $C^0([0,1])$. (Rudin RCA $\boxed{T}$ 3.9)

Minkowski's Inequality

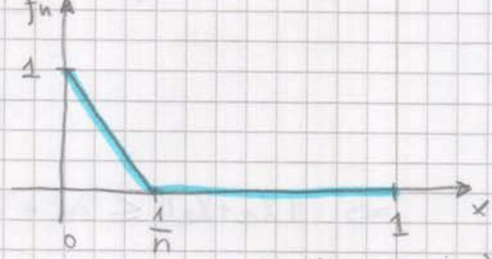
Claim: Let $(p,q)\in[1,\infty]^2$ be given s.t. $p\neq q$. Then $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent. (in the sense of Q1).

Proof: WLOG $p < q$. Then $p < \infty$.

Claim: $\nexists$ $C > 0$ s.t. $\|f\|_q \leq C\|f\|_p$ $\forall$ $f \in C^0([0,1])$

Proof: Define $f_n(x) := \begin{cases} 1 - nx & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$ $\forall$ $n \in \mathbb{N} \setminus \{0\}$.

Clearly $f_n \in C^0([0,1])$ $\forall$ $n \in \mathbb{N} \setminus \{0\}$.



Then $\|f_n\|_p \equiv \left\{ \int_0^1 |f_n(x)|^p dx \right\}^{1/p} =$

$$= \left\{ \int_0^{1/n} (1-nx)^p dx \right\}^{1/p} =$$

$$\begin{matrix} y := 1 - nx \\ dy = -n dx \end{matrix} \quad \stackrel{\curvearrowright}{=} \left\{ \int_{y=1}^{y=0} y^p \left(-\tfrac{1}{n}\right) dx \right\}^{1/p} =$$

$$= \left\{ \left(-\tfrac{1}{n}\right) \frac{1}{p+1} y^{p+1} \Big|_1^0 \right\}^{1/p} =$$

$$= \frac{1}{[n(p+1)]^{1/p}}$$

Assume $q < \infty$ as well.

Then $\frac{\|f_n\|_q}{\|f_n\|_p} = \frac{[n(p+1)]^{1/p}}{[n(q+1)]^{1/q}} = \frac{(p+1)^{1/p}}{(q+1)^{1/q}} n^{\left(\frac{1}{p} - \frac{1}{q}\right)}$

But $\frac{1}{p} - \frac{1}{q} > 0$

$\Longrightarrow \lim_{n \to \infty} \frac{\|f_n\|_q}{\|f_n\|_p} = \infty$

Next, if $q = \infty$, $\|f_n\|_\infty = 1$ (highest value).

$\Longrightarrow \lim_{n \to \infty} \frac{\|f_n\|_\infty}{\|f_n\|_p} \stackrel{\lim}{=} \sqrt{(p+1)^{1/p}}\, n^{1/p} = \infty$ also.

Thus one of the requirements for equivalence of norms cannot always be fulfilled. $\blacksquare$

Q5 Let $(p, q) \in [1, \infty]^2$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Define $\forall$ $x \in \mathbb{R}^n$, $\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

Claim: $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. (Minkowski's discrete inequality or the counting measure & Rudin RCA)

Claim: $\forall\, y \in \mathbb{R}^n$, $\|y\|_q = \sup(\{ \frac{\langle x,y\rangle}{\|x\|_p} \mid x \in \mathbb{R}^n \setminus \{0\} \})$
where $\langle x,y\rangle \equiv \sum_{i=1}^{n} x_i y_i$.

Proof: $\boxed{\leq}$ Hölder's inequality states that $\langle x,y\rangle \leq \|x\|_p \|y\|_q$.

$$\Longrightarrow \quad \frac{\langle x,y\rangle}{\|x\|_p} \leq \|y\|_q \quad \forall\ x \in \mathbb{R}^n \setminus \{0\} \qquad (*)$$

Denote the sup by $S(y)$.
Then $\forall\, \varepsilon > 0\ \exists\ x_\varepsilon \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$S(y) - \varepsilon < \frac{\langle x_\varepsilon, y\rangle}{\|x_\varepsilon\|_p} \leq S(y)$$

$$\overset{(*)}{\Longrightarrow} \quad S(y) - \varepsilon < \|y\|_q \quad \forall\ \varepsilon > 0$$

$$\overset{\varepsilon \to 0}{\Longrightarrow} \quad S(y) \leq \|y\|_q$$

In the future, abbreviate this process by saying "Take the sup() of $(*)$"

So we have one inequality.
$\boxed{\geq}$ For the other side, assume $y \neq 0$ (otherwise the relation holds easily).

Define $\sigma(\alpha) := \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha > 0 \\ -1 & \alpha < 0 \end{cases} \quad \forall\ \alpha \in \mathbb{R}$.

Define $\tilde{x}(y)$ by the components $(\tilde{x}(y))_i := \sigma(y_i)|y_i|^{q-1}\ \forall i \in \mathbb{Z}_{n+1}$

Claim: $\|y\|_q = \frac{\langle \tilde{x}(y), y\rangle}{\|\tilde{x}(y)\|_p}$

Proof:
$$\frac{\langle \tilde{x}(y), y\rangle}{\|\tilde{x}(y)\|_p} = \frac{\sum_{i=1}^{n} \sigma(y_i)|y_i|^{q-1} y_i}{\left(\sum_{i=1}^{n} |\sigma(y_i)|y_i|^{q-1}|^p\right)^{1/p}}$$

$\sigma(y_i) y_i = |y_i|$

$(q-1)p = q$

$$= \frac{(\|y\|_q)^q}{(\|y\|_q)^{q/p}} = \|y\|_q$$

But $\tilde{x}(y) \in \mathbb{R}^n \setminus \{0\}$.

$$\Longrightarrow \quad \frac{\langle \tilde{x}(y), y\rangle}{\|\tilde{x}(y)\|_p} \leq S(y)$$

$$\Longrightarrow \quad \|y\|_q \leq S(y).$$

$\blacksquare$

6 Q4 Riesz's Lemma

Let V be a normed vector space (over $\mathbb{R}$).
Recall that having a norm defines a metric
$$\tilde{d}(x,y) := \|x-y\|$$
and thus V obtains also a metric structure, and thus a topological structure as well. In particular,
$$\mathrm{Open}(V) \equiv \{ \mathcal{U} \subseteq V \mid \forall u \in \mathcal{U}\ \exists r>0 \text{ s.t. } B_r(u) \subseteq \mathcal{U} \}$$
where $B_r(u) \equiv \{ v \in V \mid \tilde{d}(u,v) < r \}$.
Recall also that $\mathrm{Closed}(V) \equiv \{ C \subseteq V \mid (V \setminus C) \in \mathrm{Open}(V) \}$.

Claim: $\mathrm{Closed}(V) = \{ C \subseteq V \mid \forall \{v_i\}_{i\in\mathbb{N}} \subseteq C \text{ s.t. } \{v_i\}_{i\in\mathbb{N}} \to v,\ v \in C \}$

(very easy to prove (from last semester) for metric sp.).

Let $W \in \mathrm{Closed}(V) \setminus \{V\}$ be a given vector subspace.
Recall this means $0_V \in W$ and $\alpha W + \beta W \subseteq W$, $\forall (\alpha,\beta) \in \mathbb{R}^2$.
Define $d(v,W) := \inf(\{ \tilde{d}(v,w) \mid w \in W \})$ $\forall v \in V$ as a map $d(\cdot, W): V \to \mathbb{R}$

(a) Claim: $[d(v,W) > 0 \iff v \notin W]$ $\forall v \in V$.

Proof: $\boxed{\Leftarrow}$ Assume $v \in W$.
Then $\tilde{d}(v,v) = 0$ and v is one of the elements in the set on which we take the infimum. Thus, $d(v,W) \leq 0$.
But by def., $d(v,W) \geq 0$. So $d(v,W) = 0$

$\boxed{\Rightarrow}$ Assume $d(v,W) = 0$.
Then $\forall \varepsilon > 0\ \exists w_\varepsilon \in W$ s.t. $0 \leq \tilde{d}(v, w_\varepsilon) < \varepsilon$
$\Rightarrow$ We may define a sequence in W which converges to v.
But W is closed $\Rightarrow v \in W$. ■

(b) Claim: $\exists v \in V$ s.t. $\|v\| = 1$ and $d(v,W) \geq 1/2$.

Proof: By assumption, $V \neq W \Rightarrow \exists v_0 \in V \setminus W$.
By (a), $d(v_0, W) > 0$.

Claim: $\exists w_0 \in W$ s.t. $\|v_0 - w_0\| \leq 2 d(v_0, W)$

Proof: As W is a vector s/sp., $0_V \in W$ and thus $W \neq \emptyset$.
Using again "the approx. property for the infimum", $\exists w_0 \in W$ s.t. $d(v_0,W) \leq \|v_0 - w_0\| < d(v_0,W) + d(v_0,W) =$

Define $v := \frac{v_0 - w_0}{\|v_0 - w_0\|}$.

Then $\|v\| = \left\| \frac{v_0 - w_0}{\|v_0 - w_0\|} \right\| = 1$.

In addition,

$$d(v, W) = d\left(\frac{v_0 - w_0}{\|v_0 - w_0\|}, W\right) =$$

$$= \inf\left(\left\{ \left\| \frac{v_0 - w_0}{\|v_0 - w_0\|} - w \right\| \;\middle|\; w \in W \right\}\right) =$$

$$= \frac{1}{\|v_0 - w_0\|} \inf\left(\left\{ \| v_0 \underbrace{- w_0 - \|v_0 - w_0\| w}_{\in W \text{ bcs. } W \text{ is a subsp.}} \| \;\middle|\; w \in W \right\}\right) =$$

$\Rightarrow$ Can change ~~of~~ variable $w \mapsto w_0 + \|v_0 - w_0\| w$

$$= \frac{d(v_0, W)}{\|v_0 - w_0\|} \geq \frac{1}{2}$$

$\blacksquare$

(c) <u>Claim</u>: $\dim(V) < \infty \iff S = \{ v \in V \mid \|v\| = 1 \} \in \mathrm{Compact}(V)$

<u>Proof</u>: $\boxed{\Longrightarrow}$ Assume $\dim(V) < \infty$.

<u>Claim</u>: WLOG $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$. (basic linear algebra)

<u>Claim</u>: All norms on $\mathbb{R}^n$ are "equivalent", in the sense of Q1.

<u>Proof</u>: Using the fact that equivalence of norms is an eq. rel. (and thus transitive), suffice to show that $\|\cdot\|$, the Euclidean norm, is eq. to any arbitrary $\|\cdot\|'$.

<u>Claim</u>: $\|\cdot\|$ and $\|\cdot\|'$ are eq. iff $\exists (c, \tilde{c}) \in (0, \infty)^2$ s.t.

$$c\|v\| \leq \|v\|' \leq \tilde{c}\|v\| \quad \forall v \in V$$

<u>Proof</u>: $\boxed{\Longrightarrow}$ $\|\cdot\|$ and $\|\cdot\|'$ are eq. $\Rightarrow \exists \tilde{\tilde{c}} > 0$ s.t. $\frac{1}{\tilde{\tilde{c}}}\|v\| \leq \|v\|' \leq \tilde{\tilde{c}}\|v\| \;\forall v \in V$.

Take $(c, \tilde{c}) = \left(\frac{1}{\tilde{\tilde{c}}}, \tilde{\tilde{c}}\right)$.

$\boxed{\Longleftarrow}$ Assume that $\exists (c, \tilde{c}) \in (0, \infty)^2$ s.t.

$c\|v\| \leq \|v\|' \leq \tilde{c}\|v\| \quad \forall v \in V$.

Then define $\tilde{\tilde{c}} := \max\left(\left\{\frac{1}{c}, \tilde{c}\right\}\right)$.

Then $\tilde{\tilde{c}} \geq \frac{1}{c} \Rightarrow \frac{1}{\tilde{\tilde{c}}} \leq c$

Define $B := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$.

Then B is compact in the top. induced by $\|\cdot\|$.

(by Heine-Borel)

Claim: $\|\cdot\|' : \mathbb{R}^n \to \mathbb{R}$ is cont. w.r.t. to the top. induced by $\|\cdot\|$.
Proof: Exercise (write in std. basis).

By the extreme value theorem, as B is cpt. and $\|\cdot\|'$ is cont., $\exists$ $\|\cdot\|'$ attains a max M and min m on B, both of which are strictly positive as $0 \notin B$.
Then let $v \in \mathbb{R}^n$ be given. Then write $v = \|v\| \hat{v}$. Then $\hat{v} \in B$.
Then $\begin{cases} \|\hat{v}\|' \leq M \\ \|\hat{v}\|' \geq m \end{cases}$
$\Longrightarrow \begin{cases} \|v\|' \leq M\|v\| \\ \|v\|' \geq m\|v\| \end{cases}$

Claim: Equivalent norms generate the same top.
Proof: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms.
Then $\exists$ $c>0$ s.t. $\frac{1}{c}\|v\|_1 \leq \|v\|_2 \leq c\|v\|_1$ $\forall v \in V$.
We need to show that $Open_1(V) = Open_2(V)$.
$\boxed{\subseteq}$ Let $U \in Open_1(V)$ be given.
Then $\forall v \in U$ $\exists$ $r_1 > 0$ s.t. $B^1_{r_1}(v) \subseteq V$.
We need to find some $r_2 > 0$ s.t. $B^2_{r_2}(v) \subseteq B^1_{r_1}(v)$.
Define $r_2 := \frac{1}{c} r_1$.
Then if $b \in B^2_{r_2}(v)$,
$\|b - v\|_2 < r_2 = \frac{1}{c} r_1$
But $\|b - v\|_2 \geq \frac{1}{c}\|b - v\|_1$
$\Rightarrow \frac{1}{c}\|b - v\|_1 < \frac{1}{c} r_1$
$\Rightarrow b \in B^1_{r_1}(v) \Longrightarrow U \in Open_2(V)$.
$\boxed{\supseteq}$ Do the same with $r_1 := c r_2$.

Observe that by norm eq. S is bounded.
Claim: $S \in Closed(V)$ in the Euclidean top.
Proof: $\|\cdot\|$ and $\|\cdot\|_2$ (the Euc. norm) are eq. Thus $\exists$ $c > 0$ s.t. $\frac{1}{c}\|x\|_2 \leq \|x\| \leq c\|x\|_2$ $\forall x \in V$.
We'll show $(V \setminus S) \in Open(V)$.
Thus let $x \in (V \setminus S)$ be given. So $\|x\| \neq 1$.
We need to find some $r > 0$ s.t. $B^2_r(x) \subseteq (V \setminus S)$.

Define $r := \frac{11}{2c}\left|\|x\|-1\right| > 0$ 9

Then if $\|x-y\|_2 < \frac{11}{2c}\left|\|x\|-1\right|$ then by norm eq. $\|x-y\| < \frac{1}{2}\left|\|x\|-1\right|$

But $\left|\|x\|-\|y\|\right| \leq \|x-y\|$ so that

$\left|\|x\|-\|y\|\right| < \frac{1}{2}\left|\|x\|-1\right|$

Then
$$\begin{aligned}\left|\|y\|-1\right| &= \left|\|y\|-\|x\|+\|x\|-1\right| \\ &\geq \left|\left|\|y\|-\|x\|\right| - \left|\|x\|-1\right|\right| \\ &= \left|\|x\|-1\right| - \underbrace{\left|\|x\|-\|y\|\right|}_{\substack{\text{smaller than } \frac{1}{2}\left|\|x\|-1\right| \\ \Rightarrow \text{smaller than } \left|\|x\|-1\right|}} \\ &> 0\end{aligned}$$

$\Rightarrow \|y\| \neq 1$

But bcs. $S \in Closed(V)$ and $S \in Bounded(V)$ then by Heine-Borel $S \in Compact(V)$

$\boxed{! \Rightarrow !}$ Assume $\dim(V) = \infty$. We'll show $S \notin Compact(V)$.
To do so, we'll use the characterization of compact in a metric space where $S \in Compact(V) \Leftrightarrow$ Every infinite seq in S must have a converging infinite subseq.

Thus we'll define a seq. in S s.t. $\|v_n - v_m\| \geq \frac{1}{2}\ \forall n \neq m$.

Define $v_1 :=$ any vector on S.

Inductively assume n vectors have been chosen.

Define $W_n := Span(\{v_i \mid i \in \mathbb{Z}_{n+1}\})$.

Then W_n is a sub vector space of finite dim.

Claim: Every finite vector space is complete.
(seen in lecture) // (linear algebra)

Claim: Every complete v. s/sp. is closed.
(linear algebra)

$\overset{(b)}{\Rightarrow} \exists\, v_{n+1} \in S$ s.t. $d(v_{n+1}, W_n) \geq 1/2$

$\Rightarrow \|v_{n+1} - v_m\| \geq 1/2 \quad \forall\, m \leq n$.

So we have defined the desired seq. $\{v_i\}$.

But clearly $\{v_i\}$ does not converge to anything, as it is not Cauchy.

$\blacksquare$