Analysis II – Colloquium of Week I – 17/2/2015

Question 1

Def: Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a real v.sp. are called **equivalent** iff $\exists\ c>0$ s.t. $\frac{1}{c}\|x\| \le \|x\|' \le c\|x\| \quad \forall x\in X$.

Claim: "Equivalence of norms" is an equivalence relation on the set of norms on X.

Proof: $\boxed{\text{Reflexivity}}$ Pick $c=$. Then .

$\boxed{\text{Symmetry}}$ If $\|\cdot\|$ is equiv. to $\|\cdot\|'$, then $\exists\ c>0$ s.t.

$$\frac{1}{c}\|x\| \le \|x\|' \le c\|x\| \quad \forall x\in X.$$

$\implies$ $\implies \Big\{$

$\boxed{\text{Transitivity}}$ Let $\|\cdot\|$ be eq. to $\|\cdot\|'$ and $\|\cdot\|'$ be eq. to $\|\cdot\|''$.

Then $\exists\ c>0,\ \tilde{c}>0$ s.t.

$$\begin{cases} \frac{1}{c}\|x\| \overset{①}{\le} \|x\|' \overset{②}{\le} c\|x\| \\ \frac{1}{\tilde{c}}\|x\|' \overset{③}{\le} \|x\|'' \overset{④}{\le} \tilde{c}\|x\|' \end{cases}$$

Q2

Let $\mathcal{U}\in[\mathrm{Open}(\mathbb{R}^n)\setminus\{\emptyset\}]\cap\mathrm{Convex}(\mathbb{R}^n)\cap\mathrm{Bounded}(\mathbb{R}^n)$ s.t.

$$[(x\in\mathcal{U}) \iff (-x\in\mathcal{U})] \quad \forall x\in\mathbb{R}^n$$

Claim: $\|x\| := \inf(\{\lambda>0 \mid \frac{x}{\lambda}\in\mathcal{U}\}) \quad \forall x\in\mathbb{R}^n$ (**Minkowski Functional**) is a norm on $\mathbb{R}^n$.

Proof: **Claim:** $\|\mathbb{R}^n\|\subseteq\mathbb{R} \implies \|\cdot\|\in\mathbb{R}^{\mathbb{R}^n}$

Proof: By def., $\|x\|$ is an inf. on a set of (positive) real numbers. The only possibility for this claim to fail then is if the set is empty, as $\inf(\emptyset)\equiv\infty$.

However, this is not possible:

Then $0\in\mathcal{U}$.

$\mathcal{U}\in\mathrm{Open}(\mathbb{R}^n) \implies \exists\ r>0$ s.t. $B_r(0)\subseteq\mathcal{U}$ where

$B_r(0) \equiv \{x \in \mathbb{R}^n \mid \|x\|_2 < r\}$ and $\|x\|_2 \equiv \sqrt{\sum_{j=1}^{n} x_j^2}$

Note that $\forall R>0$, $RB_r(0) \equiv$

$=$

$= B_{Rr}(0)$

Next, $\forall x \in \mathbb{R}^n$, $\exists$ some $R>0$ s.t. $B_{Rr}(0) \ni x$.

$\Rightarrow \|x\| < \infty \quad \forall x \in \mathbb{R}^n$.

Claim: $\|\alpha x\| = |\alpha| \|x\| \quad \forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n$.

Proof: If $\alpha = 0$ then .

. So this case works out.

If $\alpha > 0$, $\|\alpha x\| \equiv$

$\leq$

$=$

$=$

$=$

Claim: $\inf(\alpha A) = \alpha \inf(A) \quad \forall \alpha > 0, A \subseteq \mathbb{R}$.

Proof:

$=$ $\equiv |\alpha| \|x\|$.

If $\alpha < 0$, $\|\alpha x\| \equiv$

$\equiv |\alpha| \|x\|$.

<u>Claim</u>: $\|x_1+x_2\| \leq \|x_1\|+\|x_2\| \quad \forall\ (x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n$

<u>Proof</u>: Let $\varepsilon>0$ be given.

Then $\exists\ \lambda_\varepsilon^i>0$ s.t. $\left(\frac{x_i}{\lambda_\varepsilon^i}\in\mathcal{U}\right)\wedge\left[\|x_i\|\leq\lambda_\varepsilon^i<\|x_i\|+\varepsilon\right]\forall i\in\{1,2$

$\Longrightarrow \|x_1+x_2\| < \|x_1\|+\|x_2\|+2\varepsilon \quad \forall\varepsilon>0$

$\overset{\varepsilon\to 0}{\Longrightarrow}$ The claim follows.

<u>Claim</u>: $\|x\|>0 \quad \forall\ x\in\mathbb{R}^n\setminus\{0\}$.

<u>Proof</u>: Let $x\in\mathbb{R}^n\setminus\{0\}$ be given and assume $\|x\|=0$.

$\Longrightarrow \boxed{\perp}$

<u>Q3</u>

$C^0([0,1]) \equiv \left\{ f\in\mathbb{R}^{[0,1]} \;\middle|\; f \text{ is continuous} \right\}$ $\boxed{\|f\|_\infty := \sup(\{|f(x)| \mid x\in[0,1]\})}$

Let $p\in[1,\infty)$ be given.

Define $\forall\ f\in C^0([0,1])$, $\boxed{\|f\|_p :=} \left\{ \underbrace{\int_0^1 |f(x)|^p\, dx}_{\text{Riemann integral (for now} \to \text{ later will be Lebesgue)}} \right\}^{\frac{1}{p}}$

This definition is valid because

→ <u>Claim</u>: $C^0([0,1])$ is a v.sp. over $\mathbb{R}$. (easy)

→ <u>Claim</u>: $\|\cdot\|_p$ is a norm on $C^0([0,1])$. (Rudin RCA $\boxed{T}$ 3.9
Minkowski's Inequality)

<u>Claim</u>: Let $(p,q)\in[1,\infty]^2$ be given s.t. $p\neq q$. Then $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent, (in the sense of Q1).

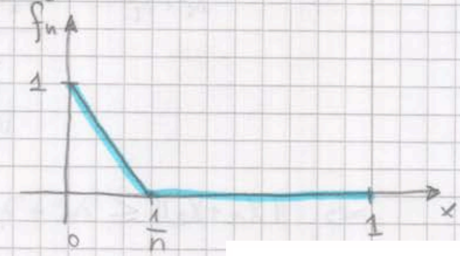
[4]

Proof: WLOG $p < q$. Then $p < \infty$.

<u>Claim</u>: $\nexists$ $C > 0$ s.t. $\|f\|_q \le C\|f\|_p$ $\forall$ $f \in C^0([0,1])$

<u>Proof</u>: Define $f_n(x) :=$ $\forall$ $n \in \mathbb{N}\setminus\{0\}$.

Clearly $f_n \in C^0([0,1])$ $\forall$ $n \in \mathbb{N}\setminus\{0\}$.



Then $\|f_n\|_p \equiv$

Assume $q < \infty$ as well.

Then $\frac{\|f_n\|_q}{\|f_n\|_p} =$

But $\frac{1}{p} - \frac{1}{q} > 0$

$\Longrightarrow \lim\limits_{n\to\infty} \frac{\|f_n\|_q}{\|f_n\|_p} = \infty$

Next, if $q = \infty$, $\|f_n\|_\infty =$

$\Longrightarrow \lim\limits_{n\to\infty} \frac{\|f_n\|_\infty}{\|f_n\|_p}$

Thus one of the requirements for equivalence of norms cannot always be fulfilled. ■

Q5 Let $(p,q) \in [1,\infty]^2$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Define $\forall$ $x \in \mathbb{R}^n$, $\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

<u>Claim</u>: $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. (Minkowski's <u>discrete</u> inequality or the counting measure & Rudin RCA)

Claim: $\forall\, y \in \mathbb{R}^n,\quad \|y\|_q = \sup\left(\left\{ \frac{\langle x,y\rangle}{\|x\|_p} \;\middle|\; x \in \mathbb{R}^n \setminus \{0\} \right\}\right) =$ 5

where $\langle x,y\rangle \equiv \sum_{i=1}^{n} x_i y_i$.

Proof: $\boxed{\leq}$ Hölder's inequality states that $\langle x,y\rangle \leq \|x\|_p \|y\|_q$.

$$\Longrightarrow \quad \frac{\langle x,y\rangle}{\|x\|_p} \leq \|y\|_q \quad \forall\, x \in \mathbb{R}^n \setminus \{0\} \quad ⊛$$

In the future, abbreviate this process by saying "Take the sup() of ⊛"

So we have one inequality.

For the other side, assume $y \neq 0$ (otherwise the relation holds easily).

Define $\sigma(\alpha) := \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha > 0 \\ -1 & \alpha < 0 \end{cases} \quad \forall\, \alpha \in \mathbb{R}$.

Define $\tilde{x}(y)$ by the components $(\tilde{x}(y))_i := \sigma(y_i)\,|y_i|^{q-1} \quad \forall\, i \in \mathbb{Z}_n^+$

Claim: $\|y\|_q = \dfrac{\langle \tilde{x}(y), y\rangle}{\|\tilde{x}(y)\|_p}$

Proof:

But $\tilde{x}(y) \in \mathbb{R}^n \setminus \{0\}$.

6 Q4 Riesz's Lemma

Let V be a normed vector space (over $\mathbb{R}$).
Recall that having a norm defines a metric

$$d(x,y) := \|x-y\|$$

and thus V obtains also a metric structure, and thus a topological structure as well. In particular,

$$\mathrm{Open}(V) \equiv \{\, \mathcal{U} \subseteq V \mid \forall u \in \mathcal{U}\ \exists r>0 \text{ s.t. } B_r(u) \subseteq \mathcal{U} \,\}$$

where $B_r(u) \equiv \{ v \in V \mid d(u,v) < r \}$.

Recall also that $\mathrm{Closed}(V) \equiv \{ C \subseteq V \mid (V \setminus C) \in \mathrm{Open}(V) \}$.

Claim: $\mathrm{Closed}(V) = \{ C \subseteq V \mid \forall \{v_i\}_{i\in\mathbb{N}} \subseteq C \text{ s.t. } \{v_i\}_{i\in\mathbb{N}} \to v,\ v \in C \}$

(Very easy to prove (from last semester) for metric sp.).

Let $W \in \mathrm{Closed}(V) \setminus \{V\}$ be a given vector subspace.
Recall this means $0_V \in W$ and $\alpha W + \beta W \subseteq W$, $\forall (\alpha,\beta) \in \mathbb{R}^2$.
Define $d(v,W) := \inf(\{ d(v,w) \mid w \in W \})$ $\forall v \in V$ as a map $d(\cdot, W) : V \to \mathbb{R}$

(a) Claim: $[d(v,W) > 0 \iff v \notin W]$ $\forall v \in V$.

Proof: $\boxed{\Leftarrow}$ Assume $v \in W$.

$\boxed{\Rightarrow}$ Assume $d(v,W) = 0$.

(b) Claim: $\exists v \in V$ s.t. $\|v\| = 1$ and $d(v,W) \geq 1/2$.

Proof: By assumption, $V \neq W \implies \exists v_0 \in V \setminus W$.
By (a), $d(v_0, W) > 0$.

Claim: $\exists w_0 \in W$ s.t. $\|v_0 - w_0\| \leq 2d(v_0, W)$

Proof:

Define $v_1 := \frac{v_0 - w_0}{\|v_0 - w_0\|}$.

Then $\|v_0\| =$.

In addition,

$d(v, W) =$

(c) Claim: $\dim(V) < \infty \iff S := \{ v \in V \mid \|v\| = 1 \} \in \mathrm{Compact}(V)$

Proof: $\boxed{\Longrightarrow}$ Assume $\dim(V) < \infty$.

Claim: WLOG $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$.
(basic linear algebra)

Claim: All norms on $\mathbb{R}^n$ are "equivalent", in the sense of Q1.

Proof: Using the fact that equivalence of norms is an eq. rel. (and thus transitive), suffice to show that $\|\cdot\|$, the Euclidean norm, is eq. to any arbitrary $\|\cdot\|'$.

Claim: $\|\cdot\|$ and $\|\cdot\|'$ are eq. iff $\exists (c, \bar{c}) \in (0, \infty)^2$ s.t.
$$c\|v\| \le \|v\|' \le \bar{c}\|v\| \quad \forall v \in V$$

Proof: $\boxed{\Longrightarrow}$

$\boxed{\Longleftarrow}$

Define $B := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$.
Then B is compact in the top. induced by $\|\cdot\|$.
(by Hei...

Claim: $\|\cdot\|' : \mathbb{R}^n \to \mathbb{R}$ is cont. w.r.t. to the top. induced by $\|\cdot\|$.

Proof: Exercise (write in std. basis).

By the , as B is cpt. and $\|\cdot\|'$ is cont., $\exists$ $\|\cdot\|'$ attains a max M and min m on B, both of which are strictly positive as $0 \notin B$.

Then let $v \in \mathbb{R}^n$ be given. Then write $v = \|v\| \hat{v}$. Then $\hat{v} \in B$.

Then

$\Longrightarrow$

Claim: Equivalent norms generate the same top.

Proof: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms. Then $\exists$ $c > 0$ s.t. $\frac{1}{c}\|v\|_1 \leq \|v\|_2 \leq c\|v\|_1$ $\forall v \in V$.

We need to show that $\text{Open}_1(V) = \text{Open}_2(V)$.

$\boxed{\subseteq}$ Let $\mathcal{U} \in \text{Open}_1(V)$ be given.

Then $\forall v \in \mathcal{U}$ $\exists$ $r_1 > 0$ s.t. $B^1_{r_1}(v) \subseteq V$.

We need to find some $r_2 > 0$ s.t. $B^2_{r_2}(v) \subseteq B^1_{r_1}(v)$.

Define $r_2 := \frac{1}{c} r_1$.

Then if $b \in B^2_{r_2}(v)$,

$\|b - v\|_2 < r_2 = \frac{1}{c} r_1$

But $\|b - v\|_2 \geq \frac{1}{c}\|b - v\|_1$

$\Longrightarrow \frac{1}{c}\|b - v\|_1 < \frac{1}{c} r_1$

$\Longrightarrow b \in B^1_{r_1}(v) \Longrightarrow \mathcal{U} \in \text{Open}_2(V)$.

$\boxed{\supseteq}$ Do the same with $r_1 := c r_2$.

Observe that by norm eq., S is bounded.

Claim: $S \in \text{Closed}(V)$ in the Euclidean top.

Proof: $\|\cdot\|$ and $\|\cdot\|_2$ (the Euc. norm) are eq. Thus $\exists$ $c > 0$ s.t. $\frac{1}{c}\|x\|_2 \leq \|x\| \leq c\|x\|_2$ $\forall x \in V$.

We'll show $(V \setminus S) \in \text{Open}(V)$.

Thus let $x \in (V \setminus S)$ be given. So .

We need to find some $r > 0$ s.t. $B^2_r(x) \subseteq (V \setminus S)$.

Define $r :=$ > 0

Then if $\|x-y\|_2 <$ then by norm

eq. $\|x-y\| <$

But $\big|\|x\|-\|y\|\big| \leq \|x-y\|$ so that

$\big|\|x\|-\|y\|\big| <$

Then $\big|\|y\|-1\big| =$

$\geq$

$=$

> 0

$\Rightarrow \|y\| \neq 1 \Rightarrow$

But bcs. $S \in Closed(V)$ and $S \in Bounded(V)$ then by Heine-Borel $S \in Compact(V)$

$\boxed{! \Rightarrow}$ Assume $\dim(V) = \infty$. We'll show $S \notin Compact(V)$.
To do so, we'll use the characterization of compact in a metric space where $S \in Compact(V) \Leftrightarrow$ Every infinite seq in S must have a converging infinite subseq.

Thus we'll define a seq. in S s.t. $\|v_n - v_m\| \geq \frac{1}{2}\ \forall n \neq m$.

Define $v_1 :=$

Inductively assume n s. vectors have been chosen.

Define $W_n :=$

Then W_n is a sub vector space of finite dim.

Claim: Every finite vector space is complete. (seen in lecture) // (linear algebra)

Claim: Every complete v. s/sp. is closed. (linear algebra)

$\overset{(b)}{\Rightarrow} \exists\, v_{n+1} \in S$ s.t. $d(v_{n+1}, W_n) \geq 1/2$

$\Rightarrow$.

So we have defined the desired seq. $\{v_i\}$.