1 Let $n \in \mathbb{N}\setminus\{0\}$.

Recall $\mathrm{Mat}_n(\mathbb{R}) \equiv$ All $n\times n$ matrices with entries in $\mathbb{R}$.

Claim: $\mathrm{Mat}_n(\mathbb{R})$ is a vector space (over $\mathbb{R}$).

Define $\|\cdot\|_2 : \mathrm{Mat}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ by $A \longmapsto \left\{ \sum_{(i,j)\in(\mathbb{Z}_n+1)^2} [(A)_{ij}]^2 \right\}^{1/2}$

where $(A)_{ij}$ is the entry on the i^{th} row and j^{th} column of the matrix A and $\mathbb{Z}_n+1 \equiv \{1,\dots,n\}$.

Claim: $\|\cdot\|_2$ is a norm on the v/sp. $\mathrm{Mat}_n(\mathbb{R})$, making $\mathrm{Mat}_n(\mathbb{R})$ "complete" w.r.t. $\|\cdot\|_2$

Recall that a map $\mathrm{Mat}_n(\mathbb{R})\times\mathrm{Mat}_n(\mathbb{R}) \longrightarrow \mathrm{Mat}_n(\mathbb{R})$ is naturally defined, called "matrix multiplication" and given by:

$(A,B) \longmapsto$ A matrix whose $(i,j)^{th}$ component is given by $\sum_{k\in\mathbb{Z}_n+1} A_{ik}B_{kj}$

Thus, $\mathrm{Mat}_n(\mathbb{R})$ is a $\underbrace{\text{normed v/sp. complete w.r.t. its norm}}_{\equiv \text{Banach space}}$ which has a certain multiplication defined on it as well.

In fact, this multiplication makes $\mathrm{Mat}_n(\mathbb{R})$ into an associative ring, as it obeys the associative ring axioms: (Herstein "Topics in Algebra" ch. 3)

1. $(\mathrm{Mat}_n(\mathbb{R}), +)$ is an Abelian group.
2. $(\mathrm{Mat}_n(\mathbb{R}),$ "matrix multiplication"$)$ is a semi-group.
3. The two operations are distributive:

$$\begin{cases} A(B+C) = AB+AC \\ (B+C)A = BA+CA \end{cases} \quad \forall\ (A,B,C)\in[\mathrm{Mat}_n(\mathbb{R})]^3$$

These statements are easily verifiable using the definitions of "matrix multiplication" and addition.

In fact, this multiplication turns $\mathrm{Mat}_n(\mathbb{R})$ into an associative algebra, as it obeys an extra axiom: (Herstein ch. 6)

4. Compatibility of "matrix mul." with v/sp. structure:

$$\alpha(AB) = (\alpha A)B = A(\alpha B) \quad \forall\ (\alpha,A,B)\in\mathbb{R}\times[\mathrm{Mat}_n(\mathbb{R})]^2$$

A Banach algebra is defined as an associative algebra which is also a Banach space obeying the additional condition that

$$\|xy\| \leq \|x\|\,\|y\| \quad \forall\ (x,y)\in(\text{Space})^2.$$

(Rudin "Functional Analysis" chapter 10)

Claim: The multiplication in a Banach algebra V is continuous.

Proof: Recall that continuity may be characterized in terms of

(1st countable: each point has a countable nbhd basis)

sequential continuity for sufficiently nice topological spaces (If the space is first countable, which metric spaces always are. V with its norm is a metric space and so V^2 is also a metric space (but with a different metric: $\forall\, p \in [1,\infty)$

$$d_p(\underbrace{(x_1,y_1)}_{\in V^2}, \underbrace{(x_2,y_2)}_{\in V^2}) := (\|x_1-x_2\|^p + \|y_1-y_2\|^p)^{1/p}$$

or

$$d_\infty((x_1,y_1),(x_2,y_2)) := \max(\{\|x_1-x_2\|, \|y_1-y_2\|\})$$

are possible metrics on V^2. $\Rightarrow$ Use seq. continuity).

So let $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ be a seq. in V^2 converging to $(x,y)\in V^2$.

We need to show the seq. $\{x_ny_n\}_{n\in\mathbb{N}}$ in V converges to $xy \in V$.

That is, $\forall\, \varepsilon>0\ \exists\, m(\varepsilon)\in\mathbb{N}$ s.t. if $n\in\mathbb{N}$ and $n \geq m(\varepsilon)$ then $\|x_ny_n - xy\| < \varepsilon$.

But we have:
$$\begin{aligned}\|x_ny_n - xy\| &= \|(x_n-x)y_n + x(y_n-y)\| \\ &\leq \|(x_n-x)y_n\| + \|x(y_n-y)\| \quad \text{norm axiom} \\ &\leq \|x_n-x\|\,\|y_n\| + \|x\|\,\|y_n-y\| \quad \text{Banach algebra}\end{aligned}$$

and we may certainly make the last expression arbitrarily small for sufficiently large n due to the convergence of $\{x_n\}\to x$ and $\{y_n\}\to y$. ∎

(a) Claim: $\mathrm{Mat}_n(\mathbb{R})$ is a Banach algebra.

Proof: Assuming all other axioms are easy, we show only the axiom regarding multiplication and the norm:

$$\|AB\|_2 \leq \|A\|_2\,\|B\|_2 \quad \forall\ (A,B)\in[\mathrm{Mat}_n(\mathbb{R})]^2.$$

$$(\|AB\|_2)^2 \equiv \sum_{(i,j)} \underbrace{\left[\sum_k (A)_{ik}(B)_{kj}\right]^2}_{\equiv |\langle a_i, b_j\rangle|^2}$$

where $a_i \in \mathbb{R}^n$ is the vector taken as the i^{th} row in A
$b_j \in \mathbb{R}^n$ is the vector taken as the j^{th} column in B

Now use the Cauchy-Schwarz inequality on the inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ to obtain $|\langle a_i, b_j\rangle|^2 \leq \|a_i\|^2\,\|b_j\|^2$ where $\|a_i\|^2 \equiv \sum_k [(a_i)_k]^2 = \sum_k [(A)_{ik}]^2$ and $\|b_j\|^2 = \sum_k [(B)_{kj}]^2$

so that we obtain:

$$(\|AB\|_2)^2 \leq \sum_{(i,j)} \left(\sum_k [(A)_{ik}]^2\right)\left(\sum_k [(B)_{kj}]^2\right) =$$

$$= \Big(\sum_{(i,k)} [(A)_{ik}]^2\Big)\Big(\sum_{(k,j)} [(B)_{kj}]^2\Big)$$

$$\equiv (\|A\|_2)^2 (\|B\|_2)^2$$

■

(b) Claim: $\|Ax\|_2 \leq \|A\|_2 \|x\|_2 \quad \forall \quad A \in Mat_n(\mathbb{R})$ and $x \in \mathbb{R}^n$.

Proof: $$(\|Ax\|_2)^2 \equiv \sum_i [(Ax)_i]^2 = \sum_i \underbrace{\Big[\sum_j (A)_{ij} (X)_j\Big]^2}_{|\langle a_i, x\rangle|^2}$$

where $a_i \in \mathbb{R}^n$ is as above. $\Rightarrow |\langle a_i, x\rangle|^2 \leq (\|a_i\|)^2 (\|x\|)^2$

Cauchy-Schwarz again on $\mathbb{R}^n$

Thus $$(\|Ax\|_2)^2 \leq \sum_i (\|a_i\|)^2 (\|x\|)^2$$

$$= \sum_i \Big(\sum_j (a_i)_j^2\Big) (\|x\|)^2$$

$$= \Big\{\sum_{(i,j)} [(A)_{ij}]^2\Big\} (\|x\|)^2 \equiv (\|A\|_2)^2 (\|x\|)^2$$

■

Q2 A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called "absolutely summable" iff the series $\{\sum_{j\leq n} x_j\}_{n\in\mathbb{N}}$ is absolutely convergent (that is, if the seq. $\{\sum_{j\leq n} |x_j|\}_{n\in\mathbb{N}}$ converges.)

Define $\|\{x_n\}_{n\in\mathbb{N}}\| := \sum_{n\in\mathbb{N}} |x_n|$ for every $\underbrace{\text{abs. summable seq. } \{x_n\}_{n\in\mathbb{N}}}_{\text{called the space } \ell^1(\mathbb{N};\mathbb{R})}$.

Claim: $(\ell^1(\mathbb{N};\mathbb{R}), \|\cdot\|)$ is a Banach space.

Proof: There are many properties to prove:

Claim: $\ell^1(\mathbb{N};\mathbb{R})$ is a v/sp. over $\mathbb{R}$. (Exercise)

Claim: $\|\cdot\|$ is a norm on $\ell^1(\mathbb{N};\mathbb{R})$. (Exercise)

Claim: $\ell^1(\mathbb{N};\mathbb{R})$ is complete w.r.t. $\|\cdot\|$.

Proof: We must show that if $\{\{x_n^{(m)}\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is Cauchy in $\ell^1(\mathbb{N};\mathbb{R})$ then it converges to some $\{x_n\}_{n\in\mathbb{N}}$.

The fact $\{\{x_n^{(m)}\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is Cauchy in $\ell^1(\mathbb{N};\mathbb{R})$ means: $\forall\ \varepsilon>0\ \exists\ \ell(\varepsilon)\in\mathbb{N}$ s.t. if $(m_1,m_2)\in\mathbb{N}^2$ and $m_1 \geq \ell(\varepsilon)$ and $m_2 \geq \ell(\varepsilon)$ then $\|\{x_n^{(m_1)}\}_{n\in\mathbb{N}} - \{x_n^{(m_2)}\}_{n\in\mathbb{N}}\| < \varepsilon$.

But $\|\{x_n^{(m_1)}\}_{n\in\mathbb{N}} - \{x_n^{(m_2)}\}_{n\in\mathbb{N}}\| =$

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$= \|\{X_n^{(m_1)} - X_n^{(m_2)}\}_{n\in\mathbb{N}}\| \equiv \sum_{n\in\mathbb{N}} |X_n^{(m_1)} - X_n^{(m_2)}| < \varepsilon$

Claim: $\{X_n^{(m)}\}_{m\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ $\forall\, n\in\mathbb{N}$.

Proof: $\forall\, n\in\mathbb{N}$ we have

$$|X_n^{(m_1)} - X_n^{(m_2)}| \leq \sum_{n'\in\mathbb{N}} |X_{n'}^{(m_1)} - X_{n'}^{(m_2)}|$$

and thus using the above properties the claim follows.

But $\mathbb{R}$ is complete. $\Rightarrow \{X_n^{(m)}\}_{m\in\mathbb{N}} \xrightarrow{m\to\infty} X_n \in \mathbb{R}$ for some X_n.

As this holds $\forall\, n\in\mathbb{N}$, this naturally defines a sequence $\{X_n\}_{n\in\mathbb{N}}$ of real numbers.

Claim: $\{X_n\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N};\mathbb{R})$.

Proof: We need to show $\sum_{n\in\mathbb{N}} |X_n| < \infty$ (as this is a series of positive numbers that would mean that the series converges.)

Recall that every Cauchy seq. is bounded.

$\Rightarrow \exists\, M>0$ s.t. $\|\{X_n^{(m)}\}_{n\in\mathbb{N}}\| \leq M \quad \forall\, m\in\mathbb{N}$.

$$\Rightarrow \sum_{n\in\mathbb{N}} |X_n^{(m)}| \leq M \quad \forall\, m\in\mathbb{N}$$

$$\Rightarrow \sum_{n\leq N} |X_n^{(m)}| \leq M \quad \forall\, m\in\mathbb{N},\ \forall\, N\in\mathbb{N}$$

$$\overset{m\to\infty}{\Longrightarrow} \sum_{n\leq N} |X_n| \leq M \quad \forall\, N\in\mathbb{N}$$

$$\overset{N\to\infty}{\Longrightarrow} \|\{X_n\}_{n\in\mathbb{N}}\| \leq M < \infty$$

Claim: $\{\{X_n^{(m)}\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}} \xrightarrow{m\to\infty} \{X_n\}_{n\in\mathbb{N}}$

Proof: We must show that $\forall\, \varepsilon>0\ \exists\, \ell(\varepsilon)\in\mathbb{N}$ s.t. if $m\in\mathbb{N}$ and $m\geq \ell(\varepsilon)$ then $\|\{X_n^{(m)}\}_{n\in\mathbb{N}} - \{X_n\}_{n\in\mathbb{N}}\| < \varepsilon$

$\|\{X_n^{(m)}\}_{n\in\mathbb{N}} - \{X_n\}_{n\in\mathbb{N}}\| \equiv \sum_{n\in\mathbb{N}} |X_n^{(m)} - X_n|$.

For any $N\in\mathbb{N}$ we have

$$\sum_{n\leq N} |X_n^{(m)} - X_n| = \lim_{q\to\infty} \sum_{n\leq N} |X_n^{(m)} - X_n^{(q)}|$$

and $\sum_{n\leq N} |X_n^{(m)} - X_n^{(q)}| \leq \|\{X_n^{(m)}\}_{n\in\mathbb{N}} - \{X_n^{(q)}\}_{n\in\mathbb{N}}\|$

Take the lim-inf of both sides (always possible)

Then the L.H.S. becomes just lim and the R.H.S. stays in general lim inf (we don't know if it converges or not) so that we have:

$$\sum_{n\leq N} |x_n^{(m)} - x_n| \leq \liminf_{q\to\infty} \|\{x_n^{(m)}\}_{n\in\mathbb{N}} - \{x_n^{(q)}\}_{n\in\mathbb{N}}\|$$

for every $N\in\mathbb{N}$. Thus in the limit $N\to\infty$:

$$\|\{x_n^{(m)}\}_{n\in\mathbb{N}} - \{x_n\}_{n\in\mathbb{N}}\| \leq \liminf_{q\to\infty} \|\{x_n^{(m)}\}_{n\in\mathbb{N}} - \{x_n^{(q)}\}_{n\in\mathbb{N}}\|$$

for every $m\in\mathbb{N}$.

The result follows by taking another $\liminf_{m\to\infty}$ of both sides. The R.H.S. goes to zero (bcs. $\{\{x_n^{(m)}\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is Cauchy). ∎

(b) Define the convolution of two elements in $\ell^1(\mathbb{N};\mathbb{R})$:

$$\{x_n\}_{n\in\mathbb{N}} * \{y_n\}_{n\in\mathbb{N}} := \{\sum_{i\leq n} x_i y_{n-i}\}_{n\in\mathbb{N}}$$

Claim: $\ell^1(\mathbb{N};\mathbb{R})$ is a Banach algebra.

Proof: There are many properties to prove:

Claim: $(\ell^1(\mathbb{N};\mathbb{R}), +, *)$ is an assoc. ring. (Exercise)

Claim: $(\ell^1(\mathbb{N};\mathbb{R}), +, *)$ is an assoc. algebra. (Exercise)

Claim: $(\ell^1(\mathbb{N};\mathbb{R}), +, *, \|\cdot\|)$ is a Banach algebra.

Proof: We must show (relying on the above claims) only that $\|\{x_n\} * \{y_n\}\| \leq \|\{x_n\}\| \|\{y_n\}\|$

Let $N\in\mathbb{N}$ be given. Then:

$$\sum_{n\leq N} |\sum_{i\leq n} x_i y_{n-i}| \leq \sum_{n\leq N}\sum_{i\leq n} |x_i y_{n-i}|$$

$$\overset{\text{only adding terms}}{\leq} \sum_{i\leq N}\sum_{j\leq N} |x_i y_j|$$

$$\leq \|\{x_n\}\| \, \|\{y_n\}\|$$

This holds for all $N\in\mathbb{N}$, so in the lim $N\to\infty$ the result follows. In particular we have also shown $\|\{x_n\} * \{y_n\}\| < \infty$ and thus $(\{x_n\} * \{y_n\}) \in \ell^1(\mathbb{N};\mathbb{R})$. So at least one point about $\ell^1(\mathbb{N};\mathbb{R})$ being a ring is taken care of (namely, closure of $*$).

6 Q3 Let $(\mathcal{A}, \|\cdot\|)$ be a real Banach algebra.

Recall that we have a map $\mathcal{A} \xrightarrow{\exp} \mathcal{A}$ given by

$$x \xmapsto{\exp} \sum_{n\in\mathbb{N}} \frac{1}{n!} x^n$$

<u>Claim</u>: exp is well-defined

<u>Proof</u>: We must show that $\left\{\sum_{n\leq N} \frac{1}{n!} x^n\right\}_{N\in\mathbb{N}}$ actually converges. Because $\mathcal{A}$ is complete, suffice to show that $\left\{\sum_{n\leq N} \frac{1}{n!} x^n\right\}_{N\in\mathbb{N}}$ is Cauchy.

Let $(N_1, N_2) \in \mathbb{N}^2$ be given s.t. $N_1 < N_2$.

Then $\left\|\sum_{n\leq N_1} \frac{1}{n!} x^n - \sum_{n\leq N_2} \frac{1}{n} x^n\right\| =$ norm axioms

$$= \left\|\sum_{n=N_1+1}^{N_2} \frac{1}{n!} x^n\right\| \leq \sum_{n=N_1+1}^{N_2} \frac{1}{n!} \|x^n\|$$

Banach Algebra

$$\leq \sum_{n=N_1+1}^{N_2} \frac{1}{n!} \|x\|^n$$

So that $\left\{\sum_{n\leq N} \frac{1}{n!} x^n\right\}_{N\in\mathbb{N}}$ is Cauchy in $\mathcal{A}$ iff $\left\{\sum_{n\leq N} \frac{1}{n} \|x\|^n\right\}_{N\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, which is of course true bcs. $\exp(\|x\|)$ converges $\forall x \in \mathcal{A}$.

Notation: $\mathcal{A}^* := \{ x\in\mathcal{A} \mid \exists\, y\in\mathcal{A} : xy = yx = e$ where e is the identity element in $\mathcal{A}$ (if one exists)$\}$

(a) <u>Claim</u>: $\exp(yxy^{-1}) = y\exp(x)y^{-1} \quad \forall\ (x,y) \in \mathcal{A}\times\mathcal{A}^*$.

<u>Proof</u>: Let $N\in\mathbb{N}$ be given.

Then $\sum_{n\leq N} \frac{1}{n!} (yxy^{-1})^n = \sum_{n\leq N} \frac{1}{n!} \underbrace{yxy^{-1}yxy^{-1} \cdots yxy^{-1}}_{n\text{-times}} =$

$$= \sum_{n\leq N} \frac{1}{n!}\, y x^n y^{-1}$$

$$= y \sum_{n\leq N} \frac{1}{n!} x^n y^{-1} \quad \text{for all } N\in\mathbb{N}.$$

In the limit $N\to\infty$ we obtain:

$$\exp(yxy^{-1}) = \lim_{N\to\infty} \left\{ y \sum_{n\leq N} \frac{1}{n!} x^n y^{-1} \right\}$$

But we have seen that multiplication in a Banach algebra is continuous, so that

$$\left[\lim_{N\to\infty} (a_N b_N)\right] = \left[\lim_{N\to\infty} a_N\right]\left[\lim_{N\to\infty} b_N\right]$$

and thus our result follows. ■

(b)(i) Claim: If $A = \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})$ then $A^m = \operatorname{diag}(a_{11}{}^m, a_{22}{}^m, \dots, a_{nn}{}^m)$.

(linear algebra)

Thus $\forall\, N \in \mathbb{N}$ we have

$$\sum_{m\le N} \frac{1}{m!} A^m = \operatorname{diag}\left(\sum_{m\le N} \frac{1}{m!}(a_{11})^m, \dots, \sum_{m\le N} \frac{1}{m!}(a_{nn})^m\right)$$

In the limit $N\to\infty$ we get:

$$\exp(A) = \lim_{N\to\infty} \operatorname{diag}\left(\sum_{m\le N} \frac{1}{m!}(a_{11})^m, \dots, \sum_{m\le N} \frac{1}{m!}(a_{nn})^m\right)$$

$$\overset{\text{Rudin PMA } \boxed{T}\ 3.4\ (a)}{=} \operatorname{diag}(e^{a_{11}}, \dots, e^{a_{nn}})$$

(ii) Diagonalize and use (a)

(iii) Compute A^2. Then A^{2k} and A^{2k+1}

$\boxed{Q4}$ Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra, and $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathbb{R}$.

Recall that to $\{a_n\}$ and $\{b_n\}$ are associated power series:

$$z \xmapsto{P} \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad z \xmapsto{Q} \sum_{n=0}^{\infty} b_n z^n$$

with corresponding radii of convergence:

$$R_a = \left[\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right]^{-1} \quad \text{and}$$

$$R_b = \left[\limsup_{n\to\infty} \sqrt[n]{|b_n|}\right]^{-1}$$

Let $R \in (0, R_a)$ be s.t. $\left(\sum_{n=1}^{\infty} |a_n| R^n\right) < R_b$.

Thus for any $|z| < R$, $z \xmapsto{Q\circ P} \left[\sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} a_m z^m\right)^n\right]$ is well defined.

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Define a new sequence by $c_n := \sum_{m=1}^{n} b_m \sum_{j_1+j_2+\dots+j_m=n} \prod_{l=1}^{n} a_{j_l}$.

Observe that $\{c_n\}_{n\in\mathbb{N}}$ gives the corresponding power series coefficients of $Q\circ P$ if we could re-arrange the summation of the two limits.

Appendix A of Banach spaces and linear op.

Claim: $$\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{L(x)} = \underbrace{\sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} a_m x^m\right)^n}_{R(x)} \quad \forall\ x \in \mathcal{A}$$

s.t. $\|x\| < R$.

Proof: $$R(x) \equiv \lim_{N\to\infty}\left[\lim_{M\to\infty} \sum_{n=0}^{N} b_n \left(\sum_{m=1}^{M} a_m x^m\right)^n\right]$$

(b) Use part (a): $P(z) = -\sum_{k=1}^{\infty} \frac{z^k}{k} = \log(1-z)$ $\quad R = 1$

$$Q(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z) \qquad R = \infty$$

Q5 $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$ for all $n \in \mathbb{N} \setminus \{0\}$

The following discussion follows Munkre §23.

Let X be a top. space.

Def.: A separation of X is two subsets $(U, V) \in [\mathrm{Open}(X) \setminus \{\emptyset\}]$
s.t.: ① $U \cap V = \emptyset$
② $U \cup V = X$

Def.: X is connected iff $\nexists$ a separation of X.

Def.: A path from $x_1 \in X$ to $x_2 \in X$ is a map
$\gamma : [0,1] \to X$ s.t. $\gamma(0) = x_1$
$\gamma(1) = x_2$
γ is continuous

Def.: X is path-connected iff $\forall (x_1, x_2) \in X \;\exists$ path γ between these two points.

Claim: If X is path-connected then X is connected.

Proof: Assume otherwise. $\Rightarrow \exists$ a seperation of X by U, V.
$(U, V) \notin \{\emptyset\}^2 \Rightarrow \exists (u, v) \in U \times V$.
X is path-connected $\Rightarrow \exists$ path γ from u to v.

Claim: The image of a connected space under a cont. map is connected (Munkres §23)

So $\gamma([0,1]) \subseteq U \;\vee\; \gamma([0,1]) \subseteq V$

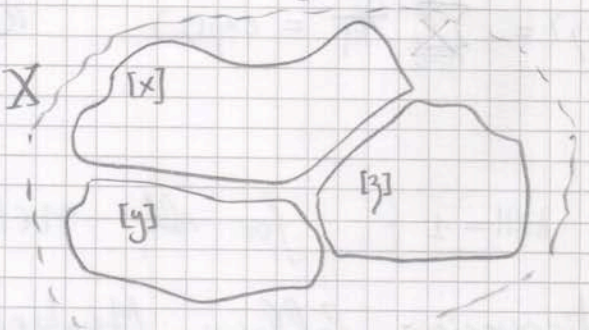
$$\gamma([0,1]) \subseteq U \Rightarrow v \in U \Rightarrow \boxed{\perp} \qquad \gamma([0,1]) \subseteq V \Rightarrow u \in V \Rightarrow \boxed{\perp}$$

Claim: If X is connected it is not necessarily path-connected.

Proof: (b)

Define an eq. rel. on X by $x \sim y$ iff $\exists A \in \mathrm{Connected}(X)$ s.t. $(x, y) \in A^2$ for all $(x, y) \in X^2$.

Then $\forall x \in X$, $[x] \equiv \{ y \in X \mid x \sim y \}$ is called the <u>connected component</u> containing x.



(a) <u>Claim</u>: $\mathbb{R}^n \setminus S^{n-1}$ has exactly two connected components $\forall \; n \geq 2$.

<u>Proof</u>: Use path-connected $\Longrightarrow$ connected. Paths are easy.

(b) Define $A := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x > 0,\, y = \sin(\tfrac{1}{x}) \right\} \cup \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y \in [-1, 1] \right\}$

<u>Claim</u>: A is connected but not path-connected.

<u>Proof</u>: $A = \underbrace{\text{closure}\Big(\underbrace{\sin\big(\tfrac{1}{(0,1]}\big)}_{\text{image of connected set under cont. function is connected}} \Big)}_{\text{closure of connected set is connected.}}$

Assume $\exists$ path from $(0,0)$ to $(1, \sin(1))$.

$\Longrightarrow$ This path would coincide with the graph of $x \mapsto \sin(\frac{1}{x})$ on $(0, 1]$

$\Longrightarrow$ The function $\sin(\frac{1}{x})$ has a limit at $x \to 0$

$\Longrightarrow \boxed{\perp}$

Def.: X is totally disconnected iff its only connected subsets are singletons.

(c) Claim: $\mathbb{Q}$ is totally disconnected.

Proof: Let $r \in \mathbb{Q}$.

Assume $[r]$ has at least two points $a < b$.

But we may pick any irrational $i \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < i < b$.

But then $(-\infty, i) \cap [r]$ and $(i, +\infty) \cap [r]$ is a separation of $[r]$ $\Rightarrow$ $[r]$ is not connected $\Rightarrow$ $\boxed{\perp}$