(Q)1 Let $n \in \mathbb{N} \setminus \{0\}$.

Recall $Mat_n(\mathbb{R}) \equiv$ All $n \times n$ matrices with entries in $\mathbb{R}$.

Claim: $Mat_n(\mathbb{R})$ is a vector space (over $\mathbb{R}$).

Define $\|\cdot\|_2 : Mat_n(\mathbb{R}) \longrightarrow \mathbb{R}$ by $A \longmapsto \left\{ \sum_{(i,j) \in (\mathbb{Z}_{n+1})^2} [(A)_{ij}]^2 \right\}^{1/2}$

where $(A)_{ij}$ is the entry on the i^{th} row and j^{th} column of the matrix A and $\mathbb{Z}_{n+1} \equiv \{1, \dots, n\}$.

Claim: $\|\cdot\|_2$ is a norm on the v/sp. $Mat_n(\mathbb{R})$, making $Mat_n(\mathbb{R})$ "complete" w.r.t. $\|\cdot\|_2$

Recall that a map $Mat_n(\mathbb{R}) \times Mat_n(\mathbb{R}) \longrightarrow Mat_n(\mathbb{R})$ is naturally defined, called "matrix multiplication" and given by:

$(A, B) \longmapsto$ A matrix whose $(i,j)^{th}$ component is given by $\sum_{k \in \mathbb{Z}_{n+1}} A_{ik} B_{kj}$

Thus, $Mat_n(\mathbb{R})$ is a $\underbrace{\text{normed v/sp. complete w.r.t. its norm}}_{\equiv \text{Banach space}}$ which has a certain multiplication defined on it as well.

In fact, this multiplication makes $Mat_n(\mathbb{R})$ into an associative ring, as it obeys the associative ring axioms: (Herstein "Topics in Algebra" ch. 3)

1. $(Mat_n(\mathbb{R}), +)$ is an Abelian group.
2. $(Mat_n(\mathbb{R}),$ "matrix multiplication"$)$ is a semi-group.
3. The two operations are distributive:

$$\begin{cases} A(B+C) = AB + AC \\ (B+C)A = BA + CA \end{cases} \quad \forall\ (A, B, C) \in [Mat_n(\mathbb{R})]^3$$

These statements are easily verifiable using the definitions of "matrix multiplication" and addition.

In fact, this multiplication turns $Mat_n(\mathbb{R})$ into an associative algebra, as it obeys an extra axiom: (Herstein ch. 6)

4. Compatibility of "matrix mul." with v/sp. structure:

$$\alpha(AB) = (\alpha A)B = A(\alpha B) \quad \forall\ (\alpha, A, B) \in \mathbb{R} \times [Mat_n(\mathbb{R})]^2$$

A Banach algebra is defined as an associative algebra which is also a Banach space obeying the additional condition that

$$\|xy\| \leq \|x\| \|y\| \quad \forall\ (x, y) \in (\text{Space})^2.$$

(Rudin "Functional Analysis" chapter 10)

Claim: The multiplication in a Banach algebra V is continuous.

Proof: Recall that continuity may be characterized in terms of

(1st countable: each point has a countable nbhd basis)

sequential continuity for sufficiently nice topological spaces (If the space is first countable, which metric spaces always are. V with its norm is a metric space and so V^2 is also a metric space (but with a different metric: $\forall\, p\in[1,\infty)$

$$d_p(\underbrace{(x_1,y_1)}_{\in V^2},\underbrace{(x_2,y_2)}_{\in V^2}) := \left(\|x_1-x_2\|^p+\|y_1-y_2\|^p\right)^{1/p}$$

or $$d_\infty((x_1,y_1),(x_2,y_2)) := \max(\{\|x_1-x_2\|,\|y_1-y_2\|\})$$

are possible metrics on V^2. $\Rightarrow$ Use seq. continuity).

So let $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ be a seq. in V^2 converging to $(x,y)\in V^2$.

We need to show the seq. $\{x_ny_n\}_{n\in\mathbb{N}}$ in V converges to $xy\in V$.

That is, $\forall\ \varepsilon>0\ \exists\ m(\varepsilon)\in\mathbb{N}$ s.t. if $n\in\mathbb{N}$ and $n\geq m(\varepsilon)$ then $\|x_ny_n-xy\|<\varepsilon$.

But we have:
$$\begin{aligned}\|x_ny_n-xy\| &= \|(x_n-x)y_n+x(y_n-y)\| \quad \text{norm axiom}\\ &\leq \|(x_n-x)y_n\|+\|x(y_n-y)\| \quad \text{Banach algebra}\\ &\leq \|x_n-x\|\,\|y_n\|+\|x\|\,\|y_n-y\|\end{aligned}$$

and we may certainly make the last expression arbitrarily small for sufficiently large n due to the convergence of $\{x_n\}\to x$ and $\{y_n\}\to y$. ∎

(a) Claim: $\mathrm{Mat}_n(\mathbb{R})$ is a Banach algebra.

Proof: Assuming all other axioms are easy, we show only the axiom regarding multiplication and the norm:

$$\|AB\|_2 \leq \|A\|_2\,\|B\|_2 \quad \forall\ (A,B)\in[\mathrm{Mat}_n(\mathbb{R})]^2.$$

$$(\|AB\|_2)^2 \equiv \sum_{(i,j)}\underbrace{\left[\sum_k (A)_{ik}(B)_{kj}\right]^2}_{\equiv|\langle a_i,b_j\rangle|^2}$$

where $a_i\in\mathbb{R}^n$ is the vector taken as the i^{th} row in A
$b_j\in\mathbb{R}^n$ is the vector taken as the j^{th} column in B

Now use the Cauchy-Schwarz inequality on the inner product space $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ to obtain $|\langle a_i,b_j\rangle|^2\leq\|a_i\|^2\,\|b_j\|^2$
where $\|a_i\|^2\equiv\sum_k[(a_i)_k]^2=\sum_k[(A)_{ik}]^2$ and $\|b_j\|^2=\sum_k[(B)_{kj}]^2$
so that we obtain:

$$(\|AB\|_2)^2 \leq$$

$$\equiv \left(\|A\|_2\right)^2 \left(\|B\|_2\right)^2$$

(b) Claim: $\|Ax\|_2 \leq \|A\|_2 \, \|x\|_2 \quad \forall \quad A \in \mathrm{Mat}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$.

Proof: $$\left(\|Ax\|_2\right)^2 \equiv \sum_i \left[(Ax)_i\right]^2 = \sum_i \underbrace{\left[\sum_j (A)_{ij} (x)_j\right]^2}_{|\langle a_i, x\rangle|^2}$$

where $a_i \in \mathbb{R}^n$ is as above. $\underset{\text{Cauchy-Schwarz again on } \mathbb{R}^n}{\Longrightarrow} |\langle a_i, x\rangle|^2 \leq \left(\|a_i\|\right)^2 \left(\|x\|\right)^2$

Thus $\left(\|Ax\|_2\right)^2 \leq$

$=$

$=$

Q2

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called "absolutely summable" iff the series $\left\{\sum_{j\leq n} x_j\right\}_{n\in\mathbb{N}}$ is absolutely convergent (that is, if the seq. $\left\{\sum_{j\leq n} |x_j|\right\}_{n\in\mathbb{N}}$ converges.)

Define $\|\{x_n\}_{n\in\mathbb{N}}\| := \sum_{n\in\mathbb{N}} |x_n|$ for every $\underbrace{\text{abs. summable seq. } \{x_n\}_{n\in\mathbb{N}}}_{\text{called the space } \ell^1(\mathbb{N};\mathbb{R})}$.

Claim: $(\ell^1(\mathbb{N};\mathbb{R}), \|\cdot\|)$ is a Banach space.

Proof: There are many properties to prove:

Claim: $\ell^1(\mathbb{N};\mathbb{R})$ is a v/sp. over $\mathbb{R}$. (Exercise)

Claim: $\|\cdot\|$ is a norm on $\ell^1(\mathbb{N};\mathbb{R})$. (Exercise)

Claim: $\ell^1(\mathbb{N};\mathbb{R})$ is complete w.r.t. $\|\cdot\|$.

Proof: We must show that if $\left\{\{x_n^{(m)}\}_{n\in\mathbb{N}}\right\}_{m\in\mathbb{N}}$ is Cauchy in $\ell^1(\mathbb{N};\mathbb{R})$ then it converges to some $\{x_n\}_{n\in\mathbb{N}}$.

The fact $\left\{\{x_n^{(m)}\}_{n\in\mathbb{N}}\right\}_{m\in\mathbb{N}}$ is Cauchy in $\ell^1(\mathbb{N};\mathbb{R})$ means:

$\forall \; \varepsilon > 0 \; \exists \; \ell(\varepsilon) \in \mathbb{N}$ s.t. if $(m_1, m_2) \in \mathbb{N}^2$ and $m_1 \geq \ell(\varepsilon)$ and $m_2 \geq \ell(\varepsilon)$ then $\left\|\{x_n^{(m_1)}\}_{n\in\mathbb{N}} - \{x_n^{(m_2)}\}_{n\in\mathbb{N}}\right\| < \varepsilon$.

But $\left\|\{x_n^{(m_1)}\}_{n\in\mathbb{N}} - \{x_n^{(m_2)}\}_{n\in\mathbb{N}}\right\| =$

$= \| \{ X_n^{(m_1)} - X_n^{(m_2)} \}_{n\in\mathbb{N}} \| \equiv \sum_{n\in\mathbb{N}} | X_n^{(m_1)} - X_n^{(m_2)} | < \varepsilon$

Claim: $\{X_n^{(m)}\}_{m\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ $\forall$ $n\in\mathbb{N}$.

Proof:

But $\mathbb{R}$ is complete. $\Longrightarrow \{X_n^{(m)}\}_{m\in\mathbb{N}} \xrightarrow{m\to\infty} X_n \in \mathbb{R}$ for some X_n.

As this holds $\forall$ $n\in\mathbb{N}$, this naturally defines a sequence $\{X_n\}_{n\in\mathbb{N}}$ of real numbers.

Claim: $\{X_n\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N};\mathbb{R})$.

Proof: We need to show $\sum_{n\in\mathbb{N}} |X_n| < \infty$ (as this is a series of positive numbers that would mean that the series converges.)

Recall that every Cauchy seq. is bounded.

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\xRightarrow{m\to\infty}$

$\xRightarrow{N\to\infty} \| \{X_n\}_{n\in\mathbb{N}} \| \leq M < \infty$

Claim: $\{\{X_n^{(m)}\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}} \xrightarrow{m\to\infty} \{X_n\}_{n\in\mathbb{N}}$

Proof: We must show that $\forall$ $\varepsilon>0$ $\exists$ $\ell(\varepsilon)\in\mathbb{N}$ s.t. if $m\in\mathbb{N}$ and $m\geq \ell(\varepsilon)$ then $\|\{X_n^{(m)}\}_{n\in\mathbb{N}} - \{X_n\}_{n\in\mathbb{N}}\| < \varepsilon$

Thus in the limit $N \to \infty$:

$$\|\{x_n^{(m)}\}_{n\in\mathbb{N}} - \{x_n\}_{n\in\mathbb{N}}\| \leq \liminf_{q\to\infty} \|\{x_n^{(m)}\}_{n\in\mathbb{N}} - \{x_n^{(q)}\}_{n\in\mathbb{N}}\|$$

for every $m\in\mathbb{N}$.
The result follows by taking another $\liminf_{m\to\infty}$ of both sides. The R.H.S. goes to zero (bcs. $\{\{x_n^{(m)}\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is Cauchy). ∎

(b) Define the convolution of two elements in $\ell^1(\mathbb{N};\mathbb{R})$:

$$\{x_n\}_{n\in\mathbb{N}} * \{y_n\}_{n\in\mathbb{N}} := \{\sum_{i\leq n} x_i y_{n-i}\}_{n\in\mathbb{N}}$$

<u>Claim</u>: $\ell^1(\mathbb{N};\mathbb{R})$ is a Banach algebra.
<u>Proof</u>: There are many properties to prove:
<u>Claim</u>: $(\ell^1(\mathbb{N};\mathbb{R}), +, *)$ is an assoc. ring. (Exercise)
<u>Claim</u>: $(\ell^1(\mathbb{N};\mathbb{R}), +, *)$ is an assoc. algebra. (Exercise)
<u>Claim</u>: $(\ell^1(\mathbb{N};\mathbb{R}), +, *, \|\cdot\|)$ is a Banach algebra.
<u>Proof</u>: We must show (relying on the above claims) only that $\|\{x_n\}*\{y_n\}\| \leq \|\{x_n\}\|\|\{y_n\}\|$
Let $N\in\mathbb{N}$ be given. Then:

$$\sum_{n\leq N} |\sum_{i\leq n} x_i y_{n-i}| \leq$$

$$\overset{\text{only adding terms}}{\leq}$$

$$\leq$$

This holds for all $N\in\mathbb{N}$, so in the lim $N\to\infty$ the result follows. In particular we have also shown $\|\{x_n\}*\{y_n\}\| < \infty$ and thus $(\{x_n\}*\{y_n\}) \in \ell^1(\mathbb{N};\mathbb{R})$. So at least one point about $\ell^1(\mathbb{N};\mathbb{R})$ being a ring is taken care of (namely, closure of $*$). ∎

6 $\boxed{\text{Q3}}$ Let $(\mathcal{A}, \|\cdot\|)$ be a real Banach algebra.

Recall that we have a map $\mathcal{A} \xrightarrow{\exp} \mathcal{A}$ given by

$$x \xmapsto{\exp} \sum_{n\in\mathbb{N}} \frac{1}{n!} x^n$$

Claim: exp is well-defined

Proof: We must show that $\left\{\sum_{n\leq N} \frac{1}{n!} x^n\right\}_{N\in\mathbb{N}}$ actually converges. Because $\mathcal{A}$ is complete, suffice to show that $\left\{\sum_{n\leq N} \frac{1}{n!} x^n\right\}_{N\in\mathbb{N}}$ is Cauchy.

Let $(N_1, N_2) \in \mathbb{N}^2$ be given s.t. $N_1 < N_2$.

Then $\left\| \sum_{n\leq N_1} \frac{1}{n!} x^n - \sum_{n\leq N_2} \frac{1}{n} x^n \right\| =$

norm axioms

$$= \left\| \sum_{n=N_1+1}^{N_2} \frac{1}{n!} x^n \right\| \leq \sum_{n=N_1+1}^{N_2} \frac{1}{n!} \|x^n\|$$

Banach Algebra

$$\leq \sum_{n=N_1+1}^{N_2} \frac{1}{n!} \|x\|^n$$

So that $\left\{\sum_{n\leq N} \frac{1}{n!} x^n\right\}_{N\in\mathbb{N}}$ is Cauchy in $\mathcal{A}$ iff $\left\{\sum_{n\leq N} \frac{1}{n} \|x\|^n\right\}_{N\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, which is of course true bcs. $\exp(\|x\|)$ converges $\forall x \in \mathcal{A}$.

Notation: $\mathcal{A}^* := \{ x \in \mathcal{A} \mid \exists y \in \mathcal{A} : xy = yx = e$ where e is the identity element in $\mathcal{A}$ (if one exists)$\}$

(a) Claim: $\exp(yxy^{-1}) = y \exp(x) y^{-1} \quad \forall (x,y) \in \mathcal{A} \times \mathcal{A}^*$.

Proof: Let $N \in \mathbb{N}$ be given.

Then $\sum_{n\leq N} \frac{1}{n!} (yxy^{-1})^n = \sum_{n\in\mathbb{N}}$ … $=$

n-times

$=$

$=$ … all $N\in\mathbb{N}$.

In the limit $N \to \infty$ we obtain:

But we have seen that multiplication in a Banach algebra is continuous, so that
$$\left[\lim_{N\to\infty}(a_N b_N)\right] = \left[\lim_{N\to\infty} a_N\right]\left[\lim_{N\to\infty} b_N\right]$$
and thus our result follows. ■

(b)(i) Claim: If $A = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$ then
$$A^m = \mathrm{diag}(a_{11}^m, a_{22}^m, \dots, a_{nn}^m).$$
(linear algebra)

Thus $\forall\, N \in \mathbb{N}$ we have
$$\sum_{m\le N} \frac{1}{m!} A^m = \mathrm{diag}\Big(\sum_{m\le N} \frac{1}{m!}(a_{11})^m, \dots, \sum_{m\le N} \frac{1}{m!}(a_{nn})^m\Big)$$

In the limit $N\to\infty$ we get:
$$\exp(A) =$$

Rudin PMA T 3.4 (a) $=$

(ii) Diagonalize and use (a)

(iii) Compute A^2. Then A^{2k} and A^{2k+1}

Q4 Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra, and $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathbb{R}$.

Recall that to $\{a_n\}$ and $\{b_n\}$ are associated power series:
$$z \xrightarrow{P} \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad z \xrightarrow{Q} \sum_{n=0}^{\infty} b_n z^n$$
with corresponding radii of convergence:
$$R_a = \left[\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right]^{-1} \quad \text{and}$$
$$R_b = \left[\limsup_{n\to\infty} \sqrt[n]{|b_n|}\right]^{-1}$$

Let $R \in (0, R_a)$ be s.t. $\left(\sum_{n=1}^{\infty} |a_n| R^n\right) < R_b$.

Thus for any $|z| < R$, $z \xrightarrow{Q\circ P} \left[\sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} a_m z^m\right)^n\right]$ is well defined.

Define a new sequence by $c_n := \sum_{m=1}^{n} b_m \sum_{j_1+j_2+\dots+j_m=n} \prod_{l=1}^{n} a_{j_l}$.

Observe that $\{c_n\}_{n\in\mathbb{N}}$ gives the corresponding power series coefficients of $Q\circ P$ if we could re-arrange the summation of the two limits.

Appendix A of Banach spaces and linear op.

Claim: $\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{L(x)} = \underbrace{\sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} a_m x^m\right)^n}_{R(x)} \quad \forall\ x \in \mathcal{A}$

s.t. $\|x\| < R$.

Proof: $R(x) \equiv \lim_{N\to\infty}\left[\lim_{M\to\infty} \sum_{n=0}^{N} b_n \left(\sum_{m=1}^{M} a_m x^m\right)^n\right]$

(b) Use part (a): $P(z) = -\sum_{k=1}^{\infty} \frac{z^k}{k} = \log(1-z) \quad R = 1$

$$Q(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z) \quad R = \infty$$

Q5 $\quad S^{n-1} \equiv \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$ for all $n \in \mathbb{N} \setminus \{0\}$

The following discussion follows Munkre §23.

Let X be a top. space.

Def.: A separation of X is two subsets $(U, V) \in [\text{Open}(X) \setminus \{\emptyset\}]$ s.t.: ① $U \cap V = \emptyset$ ② $U \cup V = X$

Def.: X is connected iff $\nexists$ a separation of X.

Def.: A path from $x_1 \in X$ to $x_2 \in X$ is a map $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x_1$, $\gamma(1) = x_2$, γ is continuous

Def.: X is path-connected iff $\forall (x_1, x_2) \in X \; \exists$ path γ between these two points.

Claim: If X is path-connected then X is connected.

Proof: Assume otherwise. $\Rightarrow \exists$ a seperation of X by U, V.

$(U, V) \notin \{\emptyset\}^2 \Rightarrow \exists (u, v) \in U \times V$.

X is path-connected $\Rightarrow \exists$ path γ from u to v.

Claim: The image of a connected space under a cont. map is connected (Munkres §23)

So $\gamma([0,1]) \subseteq U \quad \vee \quad \gamma([0,1]) \subseteq V$

$\Downarrow \qquad\qquad \Downarrow$

$v \in U \qquad\qquad u \in V$

$\Downarrow \qquad\qquad \Downarrow$

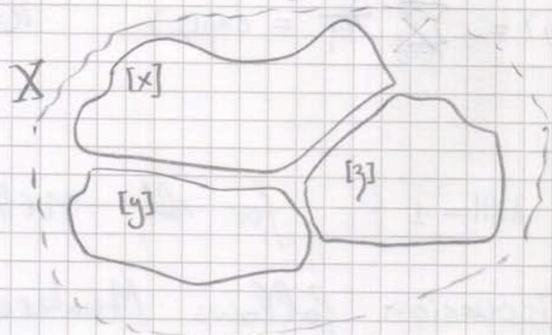
$\boxed{\perp} \qquad\qquad \boxed{\perp}$

Claim: If X is connected it is not necessarily path-connected.

Proof: (b)

Define an eq. rel. on X by $x \sim y$ iff $\exists A \in \text{Connected}(X)$ s.t. $(x, y) \in A^2$ for all $(x, y) \in X^2$.

Then $\forall\, x \in X$, $[x] \equiv \{ y \in X \mid x \sim y \}$ is called the <u>connected component</u> containing x.



(a) <u>Claim</u>: $\mathbb{R}^n \setminus S^{n-1}$ has exactly two connected components $\forall\ n \geq 2$.

<u>Proof</u>: Use path-connected $\Longrightarrow$ connected. Paths are easy.

(b) Define $A := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x > 0,\ y = \sin(\tfrac{1}{x}) \right\} \cup \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y \in [-1, 1] \right\}$

<u>Claim</u>: A is connected but not path-connected.

<u>Proof</u>: $A = \text{closure}(\quad)$

image of connected set under cont. function is connected

closure of connected set is connected.

Assume $\exists$ path from to .

$\Longrightarrow$ This path would coincide with the graph of $x \mapsto \sin(\frac{1}{x})$ on $(0, 1]$

$\Longrightarrow$ The function $\sin(\frac{1}{x})$ has a limit at $x \to 0$

$\Longrightarrow \boxed{\perp}$

<u>Def.</u>: X is totally disconnected iff its only connected subsets are singletons.

(c) <u>Claim</u>: $\mathbb{Q}$ is totally disconnected.

<u>Proof</u>: Let $r \in \mathbb{Q}$.

Assume $[r]$ has at least two points $a < b$.

But we may pick any irrational $i \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < i < b$.

But then and is a separation of $[r] \Rightarrow [r]$ is not connected $\Rightarrow \boxed{\perp}$