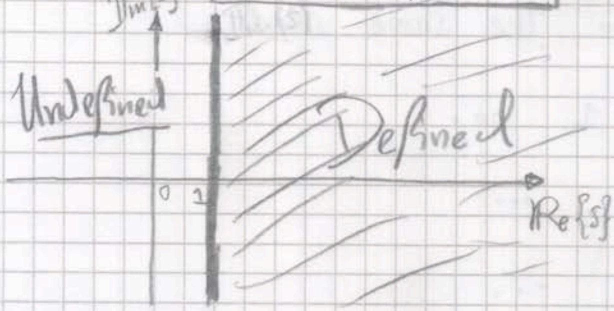
The Zeta Function

$\forall\, s \in \mathbb{C}$ s.t. $\mathrm{Re}\{s\} > 1$, define

$$\boxed{\zeta(s) := \sum_{n=1}^{\infty} n^{-s}}$$



Observe that $\{ s \in \mathbb{C} \mid \mathrm{Re}\{s\} > 1 \} \in \mathrm{Open}(\mathbb{C})$

We are interested in an analytic continuation of $\zeta(s)$, that is, a map $Z(s) : \mathbb{C} \to \mathbb{C}$ s.t. $Z(s) = \zeta(s)$ $\forall\, s \in \mathbb{C}$ s.t. $\mathrm{Re}\{s\} > 1$.
By the identity theorem of holonomic functions and the connectedness of $\mathbb{C}$, such a continuation is <u>unique</u>.

Andreas Steiger Modular Forms { "Riemann's 2nd Proof" shows that such a Z exists over the whole of $\mathbb{C}$ except at 1, where it has a simple pole with residue 1.

Moreover, it obeys the functional equation:

$$\boxed{\pi^{-s/2}\,\Gamma\!\left(\tfrac{s}{2}\right) Z(s) = \pi^{-(1-s)/2}\,\Gamma\!\left(\tfrac{1-s}{2}\right) Z(1-s)} \quad \forall\, s \in \mathbb{C} \setminus \{1\}.$$

Thus, we find, for example:

$$Z(-1) = \frac{\pi^{-1}\,\Gamma(1)\, Z(2)}{\pi^{1/2}\,\Gamma(-1/2)}$$

But $Z(2) \equiv \zeta(2)$ because $\mathrm{Re}\{2\} > 1$ and so in that area the analytic continuation must match the def. of ζ.
If you use methods of analysis I you can prove

$$\zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

$\boxed{\Gamma(1) = 1}$

Furthermore, you may also find that $\Gamma(-1/2) = -2\sqrt{\pi}$.
Thus we find $Z(-1) = \frac{\pi^{-1}\,\pi^2/6}{\sqrt{\pi}\,(-2\sqrt{\pi})} = -\frac{1}{12}$.

So this is how we get the mysterious result:

$$1+2+3+4+5+\dots = -\frac{1}{12}$$

Which actually means nothing, and is only meaningful via the link with the analytic continuation of the zeta function.

(Polchinski pp. 22)

A "silly" way to obtain the same result:

$$S_1 := 1 - 1 + 1 - 1 \dots$$

$$S_2 := 1 - 2 + 3 - 4 \dots$$

$$S := 1 + 2 + 3 + 4 \dots$$

S_1 actually diverges between 0 and 1 (depending on how many terms one sums—even or odd) so assign its value as the avg., $1/2$. $S_1 := 1/2$

$$\begin{aligned} 2S_2 &= 1 - 2 + 3 - 4 \dots \\ &\quad + 1 - 2 + 3 - 4 \dots \\ &= 1 - 1 + 1 - 1 \dots \\ &= S_1 \equiv 1/2 \end{aligned}$$

$$\Rightarrow S_2 = 1/4$$

$$\begin{aligned} S - S_2 &= 1 + 2 + 3 + 4 \dots \\ &\quad -[1 - 2 + 3 - 4 \dots] \\ &= 4 + 8 + 12 + \dots \\ &= 4(1 + 2 + 3 + \dots) \\ &= 4S \end{aligned}$$

$$\Rightarrow S - \tfrac{1}{4} = 4S \Rightarrow \boxed{S = -1/12}$$

Physics concept of regularization:

$1+2+3+\dots$ comes up in a calcula[tion]. But this is a physics calculation, so it must be finite $\Rightarrow$ Assume we derived the calculation in the wrong approach and what we really should've come up with was $\zeta(-1)$, which is indeed finite.

Why is this useful for string theory?

When calculating the energy spectrum of an open string in D dimensions, there is some constant in the energy function of the system (the Hamiltonian) given by $A = \frac{D-2}{2}\sum_{n=1}^{\infty} n$.

It turns out that in order to preserve Lorentz invariance, the value of A must be $\boxed{A=-1}$. Thus using $\zeta(-1) = -\frac{1}{12}$, we obtain the spacetime dimension of the universe, namely, $\boxed{D=26}$.