The Zeta Function

\[ \forall \operatorname{sec} C \text{ s.t. } \operatorname{Re} z > 1, \text{ define} \]

\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \text{Undefined} \quad \text{Defined} \]

\[ \Re[z] \]

Observe that \( \{ \operatorname{sec} C \mid \operatorname{Re} z > 1 \} \in \operatorname{Open}(C) \)

We are interested in an analytic continuation of \( \zeta(s) \), that is, a map \( \zeta(s) : C \to C \) s.t. \( \zeta(s) = \zeta(s) \forall \operatorname{sec} C \text{ s.t. } \operatorname{Re} z > 1 \).

By the identity theorem of holomorphic functions and the connectedness of \( C \), such a continuation is unique.

"Riemann's 2nd Proof" shows that such a \( \zeta \) exists over the whole of \( C \) except at \( 1 \), where it has a simple pole with residue 1.

Moreover, it obeys the functional equation:

\[ \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\left(s-\frac{1}{2}\right)} \Gamma\left(1-\frac{s}{2}\right) \zeta(1-s) \forall \operatorname{sec} C \setminus \{1\}. \]

Thus, we find, for example,

\[ \zeta(-1) = \frac{-\pi^{-1/2} \Gamma(1) \zeta(2)}{\pi^{1/2} \Gamma(-1/2)} \]

But \( \zeta(2) = \zeta(2) \) because \( \operatorname{Re} z > 1 \) and so in that area the analytic continuation must match the def. of \( \zeta \).

If you use methods of analysis I you can prove \( \zeta(2) = \frac{1}{\pi^2} \frac{d}{dx} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{4}{n^2} \).  

\[ \Gamma(1) = 1 \]

Furthermore, you may also find that \( \Gamma(-1/2) = -2\sqrt{\pi} \).

Thus we find \( \zeta(-1) = \frac{\pi^{-2} \pi^{3/2}}{2\sqrt{\pi}} = -\frac{1}{12} \).

So this is how we get the mysterious result:
1 + 2 + 3 + 4 + 5 + ... = -1/2

Which actually means nothing, and is only meaningful as a link with the analytic continuation of the zeta function.
(Polchinski pp. 22)

A "silly" way to obtain the same results:

\[ S_1 := 1 - 1 + 1 - 1 \ldots \]
\[ S_2 := 1 - 2 + 3 - 4 \ldots \]
\[ S' := 1 + 2 + 3 + 4 \ldots \]

\[ S_1 \text{ actually diverges between 0 and 1 } \text{(depending on how many terms one counts—even or odd)} \text{ so assign its value as the avg, } \frac{1}{2}. \quad S_1 := \frac{1}{2} \]

\[ 2S_2 = 1 - 2 + 3 - 4 \ldots + 1 - 2 + 3 - 4 \ldots = 1 - 1 + 1 - 1 \ldots = S_1 = \frac{1}{2} \]

\[ \Rightarrow S_2 = \frac{1}{4} \]

\[ S - S_2 = 1 + 2 + 3 + 4 \ldots - \left[ 1 - 2 + 3 - 4 \ldots \right] = 4 + 8 + 12 + \ldots = 4(1 + 2 + 3 + \ldots) = 4 \cdot \frac{1}{2} = 4S' \]

\[ \Rightarrow S - \frac{1}{4} = 4S' \Rightarrow S = -\frac{1}{12} \]

**Physics concept of regularization:**

\[ 1 + 2 + 3 + \ldots \]

comes up in a calculation. But this is a physics calculation, so it must be finite. \( \Rightarrow \) Assume we derived the calculation in the wrong approach and what we really should’ve come up with was \( \zeta(-1) \), which is indeed finite.

Why is this useful for string theory?

When calculating the energy spectrum of an open string in \( D \) dimensions, there is some constant in the energy function of the system (the Hamiltonian) given by \( A = \frac{D-2}{2} \sum_{n=1}^{\infty} n \).

It turns out that in order to preserve Lorentz invariance, the value of \( A \) must be \( A = -1 \). Thus using \( \zeta(-1) = -\frac{1}{12} \), we obtain the spacetime dimension of the universe, namely \( D = 26 \).