Calculus 1 — Final Solutions — May 13th 2019

Q1

1. $$\lim_{x\to\infty} \underbrace{\frac{e^x+e^{-x}}{e^x-e^{-x}}}_{\frac{1+e^{-2x}}{1-e^{-2x}}} = \frac{1+\lim_{x\to\infty} e^{-2x}}{1-\lim_{x\to\infty} e^{-2x}} = \frac{1}{1} = 1. \quad ✓$$

2. $$\lim_{x\to-\infty} \underbrace{\frac{4x^2+8x-5}{7x^2+x+9}}_{\frac{4+\frac{8}{x}-\frac{5}{x^2}}{7+\frac{1}{x}+\frac{9}{x^2}}} = \frac{4}{7}. \quad ✓$$

3. $$\lim_{x\to 0} |x|^\alpha \log(|x|) \qquad \alpha > 0$$

$$s\left(1-\frac{1}{y^s}\right) < \log(y) < \frac{1}{s}(y^s-1) \qquad \forall\, y>0, s>0$$

$$\Rightarrow s\left(1-\frac{1}{|x|^s}\right) < \log(|x|) < \frac{1}{s}(|x|^s-1)$$

$$\Rightarrow \underbrace{s|x|^\alpha\left(1-\frac{1}{|x|^s}\right)}_{|x|^\alpha-|x|^{\alpha-s}} < |x|^\alpha \log(|x|) < \frac{1}{s}\underbrace{|x|^\alpha(|x|^s-1)}_{|x|^{\alpha+s}-|x|^\alpha}$$

So if we pick $s\in(0,\alpha)$, both sides $\to 0$.

$\Rightarrow$ Using squeeze thm., $|x|^\alpha \log(|x|) \to 0$. //

Q2

1. $$\left(\frac{\sin}{\exp}\right)' = \frac{\sin'\exp-\sin\exp'}{\exp^2} = \frac{\cos\exp-\sin\exp}{\exp^2} = \frac{\cos-\sin}{\exp}. \quad //$$

2. $$\left(\operatorname{arctg}\circ\log\circ(\sqrt{\cdot}+\sin^2)\right)' = \frac{1}{1+(\log\circ(\sqrt{\cdot}+\sin^2))^2}\,\frac{1}{\sqrt{\cdot}+\sin^2}\left(\frac{1}{2\sqrt{\cdot}}+2\sin\cos\right)$$

3. $$\partial_x \int_0^x e^{-y^2}\,dy = e^{-x^2}. \quad //$$

Q3

$$f'(x) = \frac{1}{2}\log(x^2) + \frac{1}{2}x\frac{1}{x^2}2x$$

$$= \frac{1}{2}\log(x^2) + 1 \qquad (\nexists \text{ @ } x=0)$$

$$f'(x) = 0 \implies \frac{1}{2}\log(x^2) = -1$$

$$\implies x^2 = e^{-2}$$

$$\implies \boxed{x = \pm e^{-1}}$$

Note that $\frac{1}{2}\log(x^2) + 1 > 0$

$$\Updownarrow$$

$$\log(x^2) > -2$$

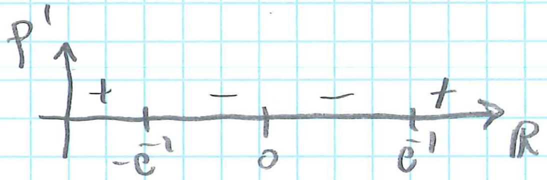
$$\Updownarrow$$

$$x^2 > e^{-2}$$

$$\Updownarrow$$

$$x \in \mathbb{R} \setminus (-e^{-1}, e^{-1})$$

$\implies$

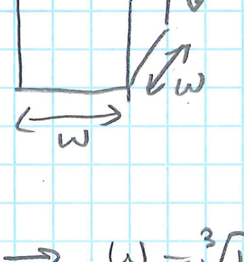


$\implies -e^{-1}$ is max,

e^{-1} is min.

Also, $f(x) \to \pm\infty$ at $x \to \pm\infty$, so these are only local.

Q4



$$V = w^2 h \qquad S = 2w^2 + 4wh$$

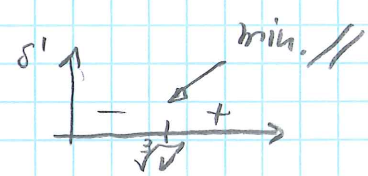
$$\implies h = \frac{V}{w^2} \qquad \implies S = 2w^2 + 4w\frac{V}{w^2}$$

$$= 2\left(w^2 + \frac{2V}{w}\right)$$

$$S' = 2\left(2w - \frac{2V}{w^2}\right) = \frac{4}{w^2}(w^3 - V)$$

$$\implies w = \sqrt[3]{V}, \quad h = \frac{V}{V^{2/3}} = V^{1-\frac{2}{3}} = \sqrt[3]{V}.$$

$$w^3 - V > 0 \iff w > \sqrt[3]{V} \implies$$



Q5

The sequence is:

$$2^{\frac{1}{2}},\ 2^{\frac{1}{2}+\frac{1}{4}},\ 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}},\ \ldots$$

$\Rightarrow$ The general term in the n^{th} place is

$$2^{\sum_{k=1}^{n}\frac{1}{2^k}} = 2^{\frac{1}{2-1}+\frac{2^{-n}}{2-1}} = 2^{1+2^{-n}}$$

Since 2^a is cont., $\lim_{n\to\infty} 2^{1+2^{-n}} = 2^{\lim_{n\to\infty} 1+2^{-n}}$

$$= 2^{1+\underbrace{\lim_{n\to\infty} 2^{-n}}_{=0}} = 2. //$$

by lecture notes Section 6.

Q6

$f(x) = \cos(x) - x$

$f'(x) = -\sin(x) - 1$

$\Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \underset{x_1=1}{=} 1 - \frac{\cos(1)-1}{-\sin(1)-1} = 1 - \frac{\frac{1}{2}-1}{-\frac{4}{5}-1} =$$

$$= 1 - \frac{-1/2}{-9/5} = 1 - \frac{5}{9\cdot 2} = \frac{18-5}{18} = \frac{13}{18}. //$$

Q7

$$\log(x) - (x+1)\log(f(x)) \overset{*}{=} \sin(\pi x) \qquad \forall \quad x>0$$

$*$ @ $x=1$

$\Rightarrow$ $-2\log(f(1)) = \sin(\pi) = 0$

$\Rightarrow \quad \log(f(1)) = 0 \Leftrightarrow \boxed{f(1)=1}.$

$\partial *$: $\quad \frac{1}{x} - \log(f(x)) - (x+1)\frac{1}{f(x)} f'(x) = \cos(\pi x)\pi$

$\partial *$ @ $x=1$: $\quad 1 - 0 - 2f'(1) = \cos(\pi)\pi = -\pi \Rightarrow \boxed{f'(1) = \frac{\pi+1}{2}}$ ✓

Q8 sin is diff. $\Longrightarrow$ can apply MVT on sin for interval $(0,x)$.

$$\overset{MVT}{\Longrightarrow} \quad \exists\, t\in(0,x): \quad \underbrace{\frac{\sin(x)-\sin(0)}{x-0}}_{\frac{\sin(x)}{x}} = \underbrace{\sin'(t)}_{\cos(t)<1}$$

$$\Longrightarrow \quad \frac{\sin(x)}{x} < 1 \quad \Longleftrightarrow \quad \sin(x) < x. \quad //$$

Q9

$$a(t) = -g \quad \Longrightarrow \quad v(t) = -gt + \underbrace{v(0)}_{=0}$$

$$\Longrightarrow \quad y(t) = -\tfrac{1}{2}gt^2 + \underbrace{y(0)}_{=h}$$

$$\Longrightarrow \quad y(t) = h - \tfrac{1}{2}gt^2 \overset{!}{=} 0$$

$$\Longleftrightarrow \quad t_0 = \pm\sqrt{\frac{2h}{g}} \quad \Longrightarrow \quad \boxed{t_0 = \sqrt{\frac{2h}{g}}}. \quad //$$

Q10 1.

$$\int_a^b \underbrace{\cot}_{\frac{\cos}{\sin}}$$

$\varphi := \sin$

$\varphi' = \cos$

$\cot = \frac{\varphi'}{\varphi}$

$$\Longrightarrow \int_a^b \cot = \int_a^b \frac{\varphi'}{\varphi} \underset{\uparrow}{=} \int_{\varphi(a)}^{\varphi(b)} \frac{1}{x}\,dx = \log\Big|_{\varphi(a)}^{\varphi(b)}$$

change of var

$$= \log(\sin(b)) - \log(\sin(a)). \quad //$$

2. $x \longmapsto \sin(x)\exp(-x^2)$ is <u>odd</u>, i.e.

$$\sin(-x)\exp(-(-x)^2) = -\sin(x)\exp(-x^2) \quad \forall\; x\geq 0.$$

$$\Longrightarrow \int_{-a}^{a} (\text{odd } f^n) = 0 \qquad (\text{here } a=2).$$

$$\Longrightarrow \int_{-2}^{2} \sin \cdot e^{-\cdot^2} = 0 \quad //.$$

3, $\int_a^b \sin = -\cos(b) + \cos(a)$

$$\int_\alpha^\beta (\cos(a) - \cos(b))\, db = \underbrace{\int_\alpha^\beta \cos(a)\, db}_{\cos(a) \underbrace{\int_\alpha^\beta db}_{(\beta-\alpha)}} - \underbrace{\int_\alpha^\beta \cos(b)\, db}_{\sin(\beta) - \sin(\alpha)}$$

$$= (\beta - \alpha)\cos(a) + \sin(\alpha) - \sin(\beta)$$

$$\int_A^B \left((\beta - \alpha)\cos(a) + \sin(\alpha) - \sin(\beta)\right) d\beta =$$

$$= \Big(\underbrace{\int_A^B \beta\, d\beta}_{\frac{1}{2}B^2 - \frac{1}{2}A^2} - \underbrace{\int_A^B \alpha\, d\beta}_{\alpha(B-A)}\Big)\cos(\alpha) + \Big(\underbrace{\int_A^B d\beta}_{B-A}\Big)\sin(\alpha) - \underbrace{\int_A^B \sin(\beta)\, d\beta}_{-\cos(B) + \cos(A)}$$

$$= \frac{1}{2}(B^2 - A^2)\cos(a) - \alpha(B-A)\cos(a) + (B-A)\sin(\alpha) + \cos(B) - \cos(A)$$

$$= \frac{1}{2}(B^2 - A^2)\cos(a) + (B-A)(\sin(\alpha) - \alpha\cos(a)) + \cos(B) - \cos(A).$$

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