Calculus 1 – Spring 2019 Section 2
Final

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Instructions

- This exam consists of ten questions (some with sub-questions). Each question is worth ten points to a total of one hundred points. Justify your answer as much as you reasonably can: the questions are open. You do *not* need to simplify your calculations.
- The exam time is from 4:10pm until 7:00pm: two hours and fifty minutes.
- *No calculators are allowed.*
- Write your UNI *without your name* clearly on *each* blue notebook you use and submit all your notebooks bundled together.
- Write clearly and legibly. Points will be taken off if the grader cannot read your answer.
- You may use any *analog* source material you wish: your notebook, prepared notes, the official lecture notes, or textbooks. You may *not* use any digital instruments, including and not limited to: smart phones, watches, laptops, tablets.

1 The exam

1.1 Exercise. Find the following limits or show that they do not exist.

1. $\lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$.
2. $\lim_{x\to-\infty}\frac{4x^2+8x-5}{7x^2+x+9}$.
3. $\lim_{x\to 0}|x|^\alpha\log(|x|)$ where $\alpha>0$ is some fixed parameter.
   Hint: $s\left(1-\frac{1}{y^s}\right)<\log(y)<\frac{1}{s}(y^s-1)$ for any $y>0$ and *any* $s>0$.

1.2 Exercise. Find the derivative of the following functions:

1. $\mathbb{R}\ni x\mapsto\frac{\sin(x)}{\exp(x)}$.
2. $(0,\infty)\ni x\mapsto\arctan\left(\log\left(\sqrt{x}+\sin(x)^2\right)\right)$.
   Hint: $\arctan'(x)=\frac{1}{1+x^2}$.
3. $(0,\infty)\ni x\mapsto\int_0^x\exp\left(-y^2\right)\mathrm{d}y$.

1.3 Exercise. Find all extrema of the function $\mathbb{R}\ni x\mapsto\begin{cases}\frac{1}{2}x\log\left(x^2\right) & x\neq 0\\ 0 & x=0\end{cases}$, decide which are global, and which are maxima or minima.

Hint: It will be useful to decide whether the function is differentiable at zero and explore its limits as $x\to\pm\infty$.

1.4 Exercise. A rectangular box *with square base* is to be manufactured so it contains some fixed given volume $v>0$ (i.e., v is a parameter of the problem). To cut on wrapping material costs, what should be the dimensions of the box (height and width (the depth is equal to the width as its base is square)) so that the surface area is *minimal*?

Hint: The answer depends on v.

1.5 Exercise. Does the sequence

$$\sqrt{2},\sqrt{2\sqrt{2}},\sqrt{2\sqrt{2\sqrt{2}}},\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\dots$$

converge? If so, to what?

Hint: $\sum_{k=1}^n\frac{1}{r^k}=\frac{1}{r-1}+\frac{r^{-n}}{r-1}$ for $r\neq 1$, $n\in\mathbb{N}$.

1.6 Exercise. Estimate the solution $x\in[0,2]$ to the equation $\cos(x)=x$.

Hint: Use *one step* of Newton's method with initial guess $x_1=1$; you may use: $\cos(1)\approx\frac{1}{2}$ and $\sin(1)\approx\frac{4}{5}$.

1.7 Exercise. Suppose $f : \mathbb{R} \to (0, \infty)$ is some differentiable function for which we don't know the explicit formula, but we do know that for all $x \in \mathbb{R}$,

$$\log(x) - (x+1)\log(f(x)) \quad = \quad \sin(\pi x)\ .$$

Find $f'(1)$ by differentiating both sides of the equation.

Hint: It might be useful to first find $f(1)$ by plugging in $x = 1$ into the above equation.

1.8 Exercise. Show that $x > \sin(x)$ for all $x > 0$.

Hint: Apply the MVT on sin on the interval $(0, x)$ and use the fact that $\mathrm{im}(\cos) \subseteq [-1, 1]$.

1.9 Exercise. A ball was dropped *at rest* from some given height $h > 0$. Calculate how long it takes for the ball to hit the ground, assuming that it is subject to *constant* acceleration at rate $g > 0$ *towards the ground*.

Hint: If $a : [0, \infty) \to \mathbb{R}$ is the acceleration as a function of time and $y : [0, \infty) \to \mathbb{R}$ is the height as a function of time, we have the basic relationship (as functions) between the two:

$$y'' = a\ .$$

1.10 Exercise. Evaluate the following integrals:

1. $\int_a^b \cot$ for some $a, b \in \mathbb{R}$ with $a < b$.

2. $\int_{-2}^{2} \sin(x) \exp\left(-x^2\right) \mathrm{d}x$.

3. $\int_A^B \left(\int_\alpha^\beta \left(\int_a^b \sin(☺)\, \mathrm{d}☺\right) \mathrm{d}b\right) \mathrm{d}\beta$ for some $a, b, \alpha, \beta, A, B \in \mathbb{R}$ such that $a < b$, $\alpha < \beta$, $A < B$.
   Hint: Work your way outwards starting from the inner-most parenthesis, carefully applying the basic rules of integration, and keeping track of the relevant integration variable at each step.