Calculus 1 – Spring 2019 Section 2–HW1 Sample Solutions

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Remark. Each "Exercise" item is one presentable unit concerning the extra credit. You only have to present *one* such exercise to get the credit, but you are encouraged to thoroughly solve all exercises.

Remark. This course is *not* meant to be an introduction to formal proof writing, but we will encounter the word 'proof' from time to time. If you are asked to give a proof, please explain intuitively in your natural language why something must be true in the clearest way you can manage. The point is to understand the reason for why things are true rather than formalize the mathematical language.

1 Naive Set Theory

Exercise 1. The set of all subsets of a set A is called its power set, and is denoted by $\mathcal{P}(A)$. List all possible subsets of the set $\{1,2,3,4\}$, i.e., write down explicitly $\mathcal{P}(A)$ as well as its size $|\mathcal{P}(A)|$. Give a formula for the total number of subsets of the set $\{1,2,3,\dots,n\}$, i.e. for $|\mathcal{P}(\{1,2,3,\dots,n\})|$ and explain how you obtained it.

Solution. To build a subset $S \subseteq A$, we must decide-for each element $a \in A$-if it shall be included in the subset S that we are building or not. That is, the subset S is a list of yes/no answers for each of the elements of A (if it should be included in S or not). If A has n elements, $|A| = n$, then we have a list of n such yes / no's, that is, there are

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

possibilities. This means that $|\mathcal{P}(\{1,2,3,4\})| = 2^4 = 16$. All possibilities are

$$\begin{aligned}\mathcal{P}(\{1,2,3,4\}) &= \{\varnothing, \{1,2,3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \\ &\quad \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \\ &\quad \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}\end{aligned}$$

You will see that when enumerating the power set $\mathcal{P}(A)$, it is useful to always get the empty set $\varnothing$ and the whole set A out of the way. Then one can proceed by the size of the subset, which is a convenient way to organize things. This leads to another formula[1]

$$2^n = \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}$$

where j is the size of the subset. This formula is a special case of Newton's binomial theorem for

$$(a+b)^n = \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} a^n b^{n-j}$$

with $a = b = 1$. Note that combinatorially $\frac{n!}{j!(n-j)!}$ represents the number of ways to pick j elements out of an n-element set.

[1] If you don't know what the $\sum$ symbol means: it is a shortcut for summation whereby we write $\sum_{j=1}^{n} a_j \equiv a_1 + a_2 + \cdots + a_n$ where $a : \mathbb{N} \to B$ is some function and we use the subscript notation a_j to denote the value of a at some $j \in \mathbb{N}$.

Exercise 2. For each of the following statements about sets, state if they are correct or incorrect, and explain your answer. If your answer is 'incorrect' provide an example, if your answer is 'correct', explain why it must be so by a step-by-step explanation. For example, the statement $A = A \cup A$ is correct. To explain this, we must decompose the statement. First we know the equality sign means that both $A \subseteq A \cup A$ and $A \cup A \subseteq A$ are true. Next, we verify each of these. The first one goes as follows. The set $A \cup A$ is the collection of all objects in either A or A (since these two are the same set, this construction is vacuous), that is, all objects in A, that is, A. So anything in A is in $A \cup A$, and vice versa, so that both statements $A \subseteq A \cup A$ and $A \cup A \subseteq A$ are indeed true.

1. $\varnothing \backslash A = \varnothing$.
2. $A \cap B = B \cap A$ and $A \cup B = B \cap A$.
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. $A \backslash (B \cup C) = (A \backslash B) \backslash C$ (this exercise had a typo previously).
5. $A \backslash (B \backslash C) = (A \backslash B) \cup C$.
6. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

Solution. We go through the statements one by one. In principle for each statement which has an equality sign, we need to show both inclusions $\subseteq$ and $\supseteq$ to establish $=$. However sometimes we suffice ourselves for a textual explanation, or just for one inclusion when the other inclusion is very similar. Also note that many times we need to negate statements which contain "and", which turns them into "or", and vice versa.

1. $\varnothing \backslash A = \varnothing$ is true. Indeed, to start verifying the statement, we would take some element from the left-hand set and show it sits inside the right-hand set, and conversely take an element inside the right-hand set and show it lies inside the left-hand set. However, there are no elements in the left-hand set nor the right-hand set: $x \in \varnothing \setminus A$ is equivalent to $x \in \varnothing$ and $x \notin A$. But $x \in \varnothing$ is false for any x. So there is nothing to verify, on either side of the equation.

2. The first statement is true: Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Said differently, $x \in B$ and $x \in A$, so that $x \in B \cap A$. Similarly, one shows that if $x \in B \cap A$ then $x \in A \cap B$. Since x was arbitrary, we conclude $A \cap B = B \cap A$.
   The second statement is false. The union of two sets is not in general equal to their intersection. Indeed, for a counter-example (there are many possible), consider $A = \varnothing$ and $B = \{\varnothing\}$ (B is not the empty set; it is rather a singleton (i.e. a set of size one) which contains the empty set). Then $A \cap B = \varnothing \cap \{\varnothing\} = \varnothing \equiv A$. The penultimate equality follows because the intersection of any set with $\varnothing$ is again $\varnothing$. However, $A \cup B = \varnothing \cup \{\varnothing\} = \{\varnothing\} \equiv B$. The penultimate equality follows because the union of any set with the empty set is that set again. Since $A \neq B$, i.e., $\varnothing \neq \{\varnothing\}$, we conclude we found an example of A, B where $A \cap B \neq A \cup B$.

3. These are the distribution laws of unions and intersections. They are both true. We show only one side of the first statement: Let $x \in A \cap (B \cup C)$. Then $x \in A$ is for sure true and also one (or both) of the following $x \in B$ or $x \in C$. Combined, this implies that one (or both) of the following holds, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$, which is precisely $x \in (A \cap B) \cup (A \cap C)$. This is however the left-hand set in the equality, so we are finished.
   There would be, in principle, the reverse inclusion to finish showing this first statement is true, and then two more inclusions for the second statement. We omit these demonstrations.

4. This statement is true. Indeed, let $x \in A \backslash (B \cup C)$. Then $x \in A$ and $x \notin (B \cup C)$, i.e. $x \in A$ and $x \notin B$ and $x \notin C$ (because negation of "or" becomes "and"). That means $x \in A \setminus B$ and $x \notin C$, which means $x \in (A \setminus B) \setminus C$. The converse inclusion follows the reverse direction.

5. This statement is false. Indeed, take $A = \{1\}$, $B = \{2\}$ and $C = \{3\}$. Then if $x = 3$, we have $x \in C$, i.e. x is in the right-hand side set. However, $3 \neq 1$ so that $x \notin A$ (note that $B \setminus C = \varnothing$ so that $A \setminus (B \setminus C) = A$).

6. This statement is true. Let $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin (B \cap C)$, which means $x \notin B$ or $x \notin C$. That is, $x \in A \setminus B$ or $x \in A \setminus C$, which is precisely the right-hand side set. The converse follows in reverse.

Exercise 3. Let X be a set and let A and B be two subsets of it, i.e., $A \subseteq X$ and $B \subseteq X$. We denote by $A^c := X \backslash A$ and $B^c := X \backslash A$. These are called the complements of A and B respectively. Prove the following two equations

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c\end{aligned}$$

by showing both directions of inclusion (i.e. $\subseteq$) for each equation. That means that there are all together four inclusions to show, but they are all similar in spirit.

Solution. These are known as De Morgan's laws. We show only the first inclusion, $(A \cup B)^c \subseteq A^c \cap B^c$, the reset being very similar to it. Let $x \in (A \cup B)^c$. Then $x \notin (A \cup B)$. That means $x \notin A$ and $x \notin B$. But this is precisely $x \in A^c \cap B^c$, which is the right-hand side set.

Exercise 4. Given two sets A and B, their *symmetric difference* is defined as

$$A \Delta B := (A \backslash B) \cup (B \backslash A)$$

(this equation had a typo previously)

Show that

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

by explaining the two inclusions encompassed in $=$. Calculate also

$$\begin{aligned}A \Delta A &= ? \\ (A \Delta B) \, \Delta \, (B \Delta C) &= ?\end{aligned}$$

Solution. We show the first inclusion $A \Delta B \subseteq (A \cup B) \setminus (A \cap B)$. Let $x \in A \Delta B$. Then by definition of $A \Delta B$, $x \in (A \backslash B) \cup (B \backslash A)$. That means that either $x \in A \setminus B$ or $x \in B \setminus A$. That is, either $x \in A$ and $x \notin B \cap A$, or $x \in B$ and $x \notin A \cap B$. Combined together (since the second statement is the same for both alternatives) this implies either $x \in A$ or $x \in B$, and, $x \notin A \cap B$, that is, $x \in A \cup B$ and $x \notin A \cap B$, which is precisely $x \in (A \cup B) \setminus (A \cap B)$. The other inclusion $A \Delta B \supseteq (A \cup B) \setminus (A \cap B)$ is very similar. Note that using the above exercises we can also simplify

$$\begin{aligned}A \Delta B &= (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \\ &= (A \cap B^c) \cup (B \cap A^c)\end{aligned}$$

Let us calculate

$$\begin{aligned}A \Delta A &\equiv (A \setminus A) \cup (A \setminus A) \\ &\quad \text{(We note that for any set, } A \setminus A = \varnothing) \\ &= \varnothing \cup \varnothing \\ &= \varnothing .\end{aligned}$$

And for the last calculation,

$$
\begin{aligned}
(A\Delta B)\,\Delta\,(B\Delta C) &= ((A\cap B^c)\cup(B\cap A^c))\,\Delta\,((B\cap C^c)\cup(C\cap B^c))\\
&= (((A\cap B^c)\cup(B\cap A^c))\cap((B\cap C^c)\cup(C\cap B^c))^c)\cup(((B\cap C^c)\cup(C\cap B^c))\cap((A\cap B^c)\cup(B\cap A^c))^c)\\
&\quad \texttt{(De-Morgan)}\\
&= (((A\cap B^c)\cup(B\cap A^c))\cap((B\cap C^c)^c\cap(C\cap B^c)^c))\cup(((B\cap C^c)\cup(C\cap B^c))\cap((A\cap B^c)^c\cap(B\cap A^c)^c))\\
&\quad \texttt{(De-Morgan again)}\\
&= (((A\cap B^c)\cup(B\cap A^c))\cap((B^c\cup C)\cap(C^c\cup B)))\cup(((B\cap C^c)\cup(C\cap B^c))\cap((A^c\cup B)\cap(B^c\cup A)))\\
&\quad \texttt{(Open brackets using distributive and associative laws, use the fact that } B^c\cap B=\varnothing\texttt{)}\\
&= (((A\cap B^c)\cup(B\cap A^c))\cap((B^c\cap C^c)\cup(C\cap B)))\cup(((B\cap C^c)\cup(C\cap B^c))\cap((A^c\cap B^c)\cup(B\cap A)))\\
&\quad \texttt{(Open brackets using distributive and associative laws, again, using the same fact)}\\
&= (((A\cap B^c)\cap(B^c\cap C^c))\cup((B\cap A^c)\cap(C\cap B)))\cup(((B\cap C^c)\cap(B\cap A))\cup((C\cap B^c)\cap(A^c\cap B^c)))\\
&\quad \texttt{(Open more brackets using associativity and use the fact that } B\cap B=B\texttt{)}\\
&= (A\cap B^c\cap C^c)\cup(B\cap A^c\cap C)\cup(C^c\cap B\cap A)\cup(C\cap B^c\cap A^c)\\
&\quad \texttt{(Use commutativity of unions and intersections to get)}\\
&= ((A\cap C^c\cap B^c)\cup(A\cap C^c\cap B))\cup((C\cap A^c\cap B)\cup(C\cap A^c\cap B^c))\\
&\quad \texttt{(Use the fact that for any set } X\texttt{, } X=(B\cup B^c)\cap X=(X\cap B)\cup(X\cap B^c)\texttt{)}\\
&= (A\cap C^c)\cup(C\cap A^c)\\
&= A\Delta C
\end{aligned}
$$

In hindsight perhaps this last calculation was a bit excessive... A student graciously pointed out that this could also be solved via the associative law for the symmetric difference, which has to be proven by itself.

2 Functions

Exercise 5. Show that for $f : A \to B$ the following holds:

1. It is injective if and only if it has a left inverse, if and only if for any $a, \tilde{a} \in A$ such that $f(a) = f(\tilde{a})$ we have $a = \tilde{a}$[2].

2. It is surjective if and only if it has a right inverse, if and only if for any $b \in B$, there is some $a \in A$ such that $f(a) = b$.

3. Show that if $f : A \to B$ is injective and $g : B \to C$ is injective then $g \circ f : A \to C$ is injective, but conversely if all we know is that $g \circ f : A \to C$ is injective, then f is injective but g need not be (i.e. provide a concrete example for this latter scenario).

Solution. We start with the items in the order they are presented. In the first two items, $f : A \to B$ is a given function.

1. Assume that f is injective. Using our definition from the lecture notes that means that for any element $b \in B$, $\left|f^{-1}(\{b\})\right| \le 1$. That means that for every $b \in B$, we have at most one element in its pre-image. Pick any $a_0 \in A$ at random. Define a new function $g : B \to A$ with the following rule: For any $b \in B$, if $\left|f^{-1}(\{b\})\right| = 0$, i.e., if that b is not covered by f, then send it to a_0. Otherwise, send it to that (unique) element in its pre-image. In a formula, this means

$$g(b) := \begin{cases} a_0 & f^{-1}(\{b\}) = \varnothing \\ a \texttt{ such that } a \in f^{-1}(\{b\}) & f^{-1}(\{b\}) \neq \varnothing \end{cases} \qquad (b \in B)$$

Note that this is a valid function. That is, g maps any element $b \in B$ to exactly one element of A. Indeed, this is guaranteed by the injectivity condition $\left|f^{-1}(\{b\})\right| \le 1$.

[2] This last condition means that knowledge of equality after applying f implies knowledge of equality before applying f, so that f "preserves" information in a certain sense.

We claim that g is the left-inverse of f. That means we want to show that $g \circ f = \mathbb{1}_A$. So let $a \in A$. Then $(g \circ f)(a) \equiv g(f(a))$. Note that $f(a) \in B$ by definition, and also that by definition, $a \in f^{-1}(\{f(a)\})$ (because $f(a) \in \{f(a)\}$!), so that $f^{-1}(\{f(a)\}) \neq \varnothing$. Due to injectivity, a is the only element in $f^{-1}(\{f(a)\})$. This means that $g(f(a)) = a$, i.e. $g \circ f = \mathbb{1}_A$.
Now assume that f has a left-inverse $g : B \to A$, and let $a, \tilde{a} \in A$ such that $f(a) = f(\tilde{a})$. Let us now apply the left-inverse g on this equation: by definition of the left inverse,

$$
\begin{aligned}
a &= (\mathbb{1}_A)(a) \\
&= (g \circ f)(a) \\
&\equiv g(f(a)) \\
&\quad \text{(By assumption that } f(a) = f(\tilde{a})\text{)} \\
&= g(f(\tilde{a})) \\
&= (g \circ f)(\tilde{a}) \\
&= (\mathbb{1}_A)(\tilde{a}) \\
&= \tilde{a}
\end{aligned}
$$

so we obtain the last condition. Now let us show it is equal to the first condition. So let $b \in B$ be given. We want to show that $\left|f^{-1}(\{b\})\right| \leq 1$. If $f^{-1}(\{b\}) = \varnothing$ then we're finished. Otherwise, we have some $a \in A$ such that $f(a) = b$. Now assume that $\left|f^{-1}(\{b\})\right| \leq 1$ were false. That means there must be some $\tilde{a} \in A \setminus \{a\}$ such that $f(\tilde{a}) = b$. But that means $f(a) = f(\tilde{a})$, and our hypothesis now says this implies $a = \tilde{a}$, i.e. a contradiction. So it must be that $\left|f^{-1}(\{b\})\right| \leq 1$ is true.

2. Let us assume that f is surjective, which by our definition mean that $f(A) = B$. We need to demonstrate that f has a right-inverse. Since any point $b \in B$ is covered by f, that means that for any $b \in B$, $f^{-1}(\{b\}) \neq \varnothing$. Define $g : B \to A$ by arbitrarily picking (once and for all), for any $b \in B$, any point $a \in f^{-1}(\{b\})$ and setting $g(b) := a$. This results in a valid, well-defined function. It also happens to be a right-inverse for f by definition. Indeed, $(f \circ g)(b) = f(g(b)) = b$ because we picked $g(b) \in f^{-1}(\{b\})$.
Now for the last condition, let us take any $b \in B$. Because there is a right-inverse, apply it on b to get $g(b) \in A$. Because it is the right inverse, that means $f(g(b)) = b$.
To show that this implies $f(A) = B$, note first that since $f(A) \subseteq B$ by definition, we only need to show $B \subseteq f(A)$. But $B \subseteq f(A)$ is precisely what we have just shown, since $f(A)$ is the set of elements in B which are covered by f.

3. Let $f : A \to B$ and $g : B \to C$ both be injective. We'd like to show that $g \circ f : A \to C$ is also injective. So let $a, \tilde{a} \in A$ be given such that $(g \circ f)(a) = (g \circ f)(\tilde{a})$. We want $a = \tilde{a}$. Our assumption means we have $g(f(a)) = g(f(a))$, i.e. we have $f(a), f(\tilde{a}) \in B$ with $g(f(a)) = g(f(\tilde{a}))$, so the injectivity of g alone implies $f(a) = f(\tilde{a})$. But now the injectivity of f implies $a = \tilde{a}$, and we're finished.
Next, let $f : A \to B$ and $g : B \to C$ be given such that $(g \circ f) : A \to C$ is injective. We first want to show this means f must be injective. So let $a, \tilde{a} \in A$ and assume that $f(a) = f(\tilde{a})$. We want to show $a = \tilde{a}$. Since $f(a) = f(\tilde{a})$, we can apply g on this equation to get $g(f(a)) = g(f(\tilde{a}))$. But now the injectivity of $g \circ f$ implies $a = \tilde{a}$ and so f is indeed injective.
Finally, in this latter scenario, we'd like to find an example where f is injective, $g \circ f$ is injective, but g is not injective. Pick

$$
\begin{aligned}
f : [0, \infty) &\to \mathbb{R} \\
x &\mapsto \sqrt{x}
\end{aligned}
$$

and

$$
\begin{aligned}
g : \mathbb{R} &\to [0, \infty) \\
x &\mapsto x^2
\end{aligned}
$$

Then $g \circ f = \mathbb{1}_{[0,\infty)}$ which is injective, and f is certainly injective, but g is not injective, because both 5 and -5 lead to 25!

Exercise 6. Show that if $f : A \to B$ and $|A| = |B| < \infty$ then f is injective if and only if it is surjective.

Solution. The key point in this exercise is that $|A| = |B| < \infty$. This in turn implies that there is a perfect match between the number of elements in A and in B. This in turn implies that if f is injective (so that no two elements of B are covered twice), since every element of A must go somewhere in B, all of B must be covered, that is, f is surjective. Conversely, if f is surjective, all elements of B are covered, but since there are just as many elements of A, there aren't any "extra" elements of A to cover an element of B twice so as to prevent f from being surjective. This is sometimes called the pigeon-hole principle.

Exercise 7. Define a bijection $f : \mathbb{N} \to 2\mathbb{N}$, where by $2\mathbb{N}$ we mean the set $\{ 2n \mid n \in \mathbb{N} \} = \{ 2, 4, 6, \dots \}$, and calculate the (unique left, which is also the right) inverse of f. Define a bijection $f : \mathbb{N} \to \mathbb{Z}$ and calculate the inverse.

Solution. Let us define

$$\begin{aligned} f : \mathbb{N} &\to 2\mathbb{N} \\ n &\mapsto 2n \end{aligned}$$

First of all this is a well-defined map, since each element of $\mathbb{N}$ gets mapped to exactly one element of $2n$ due to having an exact formula, which defines always an element of the co-domain. Define an inverse (it will turn out to be both a left and right inverse)

$$\begin{aligned} g : 2\mathbb{N} &\to \mathbb{N} \\ n &\mapsto \frac{n}{2} \end{aligned}$$

First of all this formula needs to be justified. Any element $n \in 2\mathbb{N}$ is of the form $n = 2m$ for some $m \in \mathbb{N}$. Hence, $\frac{n}{2} = m$ makes sense as an element in $\mathbb{N}$ and the formula gives a well-defined function. It is indeed both the left and right inverse of f: for any $n \in \mathbb{N}$ we have

$$\begin{aligned} (g \circ f)(n) &= \frac{f(n)}{2} \\ &= \frac{2n}{2} \\ &= n \\ &= \mathbb{1}_{\mathbb{N}}(n) \end{aligned}$$

and for any $n \in 2\mathbb{N}$ we have

$$\begin{aligned} (f \circ g)(n) &= 2g(n) \\ &= 2\frac{n}{2} \\ &= n \\ &= \mathbb{1}_{2\mathbb{N}}(n) \end{aligned}$$

so this is indeed a right and left inverse, and hence f is a bijection. Note that intuitively we "feel" that $2\mathbb{N}$ holds half as many elements as $\mathbb{N}$, but since we could setup a bijection, the two sets are commensurate in a certain way (in mathematics they are said to be of the same cardinality, i.e. the magnitude of their infinite size is the same, so to speak).

Next, let us seek a bijection $f : \mathbb{N} \to \mathbb{Z}$. First we define an auxiliary bijetion $\tilde{f} : \mathbb{N} \to \{ 0 \} \cup \mathbb{N}$ by just shifting all elements to the left:

$$\tilde{f}(n) := n - 1$$

This is indeed a bijection with the inverse $n \mapsto n+1$ (verify this). Next we define a bijection $\tilde{\tilde{f}} : \{ 0 \} \cup \mathbb{N} \to \mathbb{Z}$ via the formula

$$n \mapsto \begin{cases} 0 & n = 0 \\ \frac{n+1}{2} & n \in 2\mathbb{N} + 1 \\ -\frac{n}{2} & n \in 2\mathbb{N} \end{cases}$$

The formula gives us

$$\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 1 \\ 2 &\mapsto -1 \\ 3 &\mapsto 2 \\ 4 &\mapsto -2 \\ 5 &\mapsto 3 \\ &\dots \end{aligned}$$

This gives us all together the function $f := \tilde{\tilde{f}} \circ \tilde{f}$:

$$f(n) := \begin{cases} \frac{n}{2} & n \in 2\mathbb{N} \\ -\frac{n-1}{2} & n \in 2\mathbb{N} - 1 \end{cases}$$

One can verify that its inverse is given by $g : \mathbb{Z} \to \mathbb{N}$ with the formula

$$g(n) := \begin{cases} 2n & n > 0 \\ -2n+1 & n \leq 0 \end{cases}.$$

Exercise 8. For each of the following functions, determine if it is monotone increasing, monotone decreasing, neither or both. Also determine if the qualifier 'strictly' is appropriate. If it is neither increasing nor decreasing, determine if one could restrict the function to a subdomain such that there, it is monotone.

1. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$.
2. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ (the absolute value, defined in the lecture notes).
3. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$.
4. $f : \mathbb{R} \setminus \{ 0 \} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$.
5. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto -x$.

Solution. We go through the functions in the order they are presented:

1. For $x \mapsto x^2$ from $\mathbb{R} \to \mathbb{R}$, the function is not monotone since for negative values of x it is (strictly) decreasing yet for positive values it is (strictly) increasing. This also suggests how to restricts its domain so as to get a monotone function.
2. For the absolute value the story is essentially the same.
3. The constant function of constant zero is not monotone. But if we restrict its domain to a singleton then it is (vacuously) monotone, which is cheating.
4. This function is strictly monotone decreasing. Indeed, let $x, y \in \mathbb{R}$ such that $x < y$. Then we get $\frac{1}{x} > \frac{1}{y}$ (because taking the reciprocal flips the direction of the inequality) so that $f(x) > f(y)$.
5. This function is strictly monotone decreasing as well. Indeed, if $x < y$, then multiplying the inequality by -1 gives $-x > -y$ (the direction flips by multiplying by -1) so that $f(x) > f(y)$.