

Calculus 1 – Spring 2019 Section 2–HW1

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Remark. Each “Exercise” item is one presentable unit concerning the extra credit. You only have to present *one* such exercise to get the credit, but you are encouraged to thoroughly solve all exercises.

Remark. This course is *not* meant to be an introduction to formal proof writing, but we will encounter the word ‘proof’ from time to time. If you are asked to give a proof, please explain intuitively in your natural language why something must be true in the clearest way you can manage. The point is to understand the reason for why things are true rather than formalize the mathematical language.

1 Naive Set Theory

Exercise 1. The set of all subsets of a set A is called its power set, and is denoted by $\mathcal{P}(A)$. List all possible subsets of the set $\{1, 2, 3, 4\}$, i.e., write down explicitly $\mathcal{P}(A)$ as well as its size $|\mathcal{P}(A)|$. Give a formula for the total number of subsets of the set $\{1, 2, 3, \dots, n\}$, i.e. for $|\mathcal{P}(\{1, 2, 3, \dots, n\})|$ and explain how you obtained it.

Exercise 2. For each of the following statements about sets, state if they are correct or incorrect, and explain your answer. If your answer is ‘incorrect’ provide an example, if your answer is ‘correct’, explain why it must be so by a step-by-step explanation. For example, the statement $A = A \cup A$ is correct. To explain this, we must decompose the statement. First we know the equality sign means that both $A \subseteq A \cup A$ and $A \cup A \subseteq A$ are true. Next, we verify each of these. The first one goes as follows. The set $A \cup A$ is the collection of all objects in either A or A (since these two are the same set, this construction is vacuous), that is, all objects in A , that is, A . So anything in A is in $A \cup A$, and vice versa, so that both statements $A \subseteq A \cup A$ and $A \cup A \subseteq A$ are indeed true.

1. $\emptyset \setminus A = \emptyset$.
2. $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. $A \setminus (B \cup C) = (A \setminus B) \setminus C$ (this exercise had a typo previously).
5. $A \setminus (B \setminus C) = (A \setminus B) \cup C$.
6. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Exercise 3. Let X be a set and let A and B be two subsets of it, i.e., $A \subseteq X$ and $B \subseteq X$. We denote by $A^c := X \setminus A$ and $B^c := X \setminus B$. These are called the complements of A and B respectively. Prove the following two equations

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c\end{aligned}$$

by showing both directions of inclusion (i.e. \subseteq) for each equation. That means that there are all together four inclusions to show, but they are all similar in spirit.

Exercise 4. Given two sets A and B , their *symmetric difference* is defined as

$$A \Delta B := (A \setminus B) \cup (B \setminus A)$$

(this equation had a typo previously)

Show that

$$A\Delta B = (A \cup B) \setminus (A \cap B)$$

by explaining the two inclusions encompassed in $=$. Calculate also

$$\begin{aligned} A\Delta A &= ? \\ (A\Delta B)\Delta(B\Delta C) &= ? \end{aligned}$$

2 Functions

Exercise 5. Show that for $f : A \rightarrow B$ the following holds:

1. It is injective if and only if it has a left inverse, if and only if for any $a, \tilde{a} \in A$ such that $f(a) = f(\tilde{a})$ we have $a = \tilde{a}$ ¹.
2. It is surjective if and only if it has a right inverse, if and only if for any $b \in B$, there is some $a \in A$ such that $f(a) = b$.
3. Show that if $f : A \rightarrow B$ is injective and $g : B \rightarrow C$ is injective then $g \circ f : A \rightarrow C$ is injective, but conversely if all we know is that $g \circ f : A \rightarrow C$ is injective, then f is injective but g need not be (i.e. provide a concrete example for this latter scenario).

Exercise 6. Show that if $f : A \rightarrow B$ and $|A| = |B| < \infty$ then f is injective if and only if it is surjective.

Exercise 7. Define a bijection $f : \mathbb{N} \rightarrow 2\mathbb{N}$, where by $2\mathbb{N}$ we mean the set $\{2n \mid n \in \mathbb{N}\} = \{2, 4, 6, \dots\}$, and calculate the (unique left, which is also the right) inverse of f . Define a bijection $f : \mathbb{N} \rightarrow \mathbb{Z}$ and calculate the inverse.

Exercise 8. For each of the following functions, determine if it is monotone increasing, monotone decreasing, neither or both. Also determine if the qualifier 'strictly' is appropriate. If it is neither increasing nor decreasing, determine if one could restrict the function to a subdomain such that there, it is monotone.

1. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$.
2. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$ (the absolute value, defined in the lecture notes).
3. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 0$.
4. $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}$.
5. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto -x$.

¹This last condition means that knowledge of equality after applying f implies knowledge of equality before applying f , so that f "preserves" information in a certain sense.