Homework 10 Solution

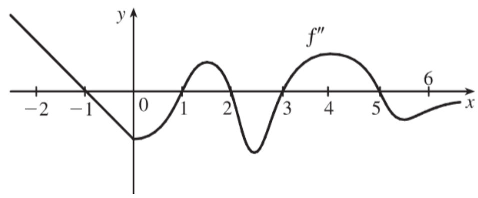
Part 1

1.

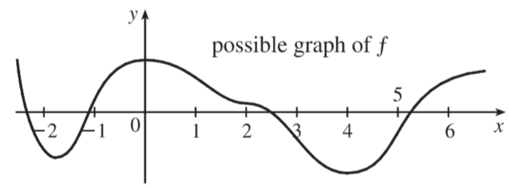
(a) Using the Test for Monotonic Functions we know that f is increasing on $(-2,0)$ and $(4,\infty)$ because $f' > 0$ on $(-2,0)$ and $(4,\infty)$, and that f is decreasing on $(-\infty,-2)$ and $(0,4)$ because $f' < 0$ on $(-\infty,-2)$ and $(0,4)$.

(b) Using the First Derivative Test, we know that f has a local maximum at $x=0$ because f' changes from positive to negative at $x=0$, and that f has a local minimum at $x=4$ because f' changes from negative to positive at $x=4$.

(c)



(d)



2.

$f(x) = x\sqrt{1-x}$, $[-1,1]$. $f'(x) = x\cdot\frac{1}{2}(1-x)^{-1/2}(-1) + (1-x)^{1/2}(1) = (1-x)^{-1/2}\left[-\frac{1}{2}x + (1-x)\right] = \dfrac{1-\frac{3}{2}x}{\sqrt{1-x}}$.

$f'(x) = 0 \;\Rightarrow\; x = \frac{2}{3}$. $f'(x)$ does not exist $\;\Leftrightarrow\; x = 1$. $f'(x) > 0$ for $-1 < x < \frac{2}{3}$ and $f'(x) < 0$ for $\frac{2}{3} < x < 1$, so $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{9}\sqrt{3}$ [≈ 0.38] is a local maximum value. Checking the endpoints, we find $f(-1) = -\sqrt{2}$ and $f(1) = 0$. Thus, $f(-1) = -\sqrt{2}$ is the absolute minimum value and $f\left(\frac{2}{3}\right) = \frac{2}{9}\sqrt{3}$ is the absolute maximum value.

$f(x) = \dfrac{3x-4}{x^2+1}$, $[-2,2]$. $f'(x) = \dfrac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2} = \dfrac{-(3x^2-8x-3)}{(x^2+1)^2} = \dfrac{-(3x+1)(x-3)}{(x^2+1)^2}$.

$f'(x) = 0 \;\Rightarrow\; x = -\frac{1}{3}$ or $x = 3$, but 3 is not in the interval. $f'(x) > 0$ for $-\frac{1}{3} < x < 2$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{3}$, so $f\left(-\frac{1}{3}\right) = \frac{-5}{10/9} = -\frac{9}{2}$ is a local minimum value. Checking the endpoints, we find $f(-2) = -2$ and $f(2) = \frac{2}{5}$. Thus, $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$ is the absolute minimum value and $f(2) = \frac{2}{5}$ is the absolute maximum value.

$f(x) = \sqrt{x^2+x+1}$, $[-2,1]$. $f'(x) = \frac{1}{2}(x^2+x+1)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$. $f'(x) = 0 \;\Rightarrow\; x = -\dfrac{1}{2}$.

$f'(x) > 0$ for $-\frac{1}{2} < x < 1$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{2}$, so $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is a local minimum value. Checking the endpoints, we find $f(-2) = f(1) = \sqrt{3}$. Thus, $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is the absolute minimum value and $f(-2) = f(1) = \sqrt{3}$ is the absolute maximum value.

Part 2

2.

Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e).$$

Let $u = 1 + 2x$, so $x = \frac{1}{2}(u-1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\begin{aligned}\int_0^4 \frac{x\,dx}{\sqrt{1+2x}} &= \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\,du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 = \tfrac{1}{4}\cdot\tfrac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9 \\ &= \tfrac{1}{6}[(27-9) - (1-3)] = \tfrac{20}{6} = \tfrac{10}{3}\end{aligned}$$

Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

Let $u = \sin^{-1} x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

Let $u = e^z + z$, so $du = (e^z + 1)\,dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z + 1}{e^z + z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{e+1} = \ln|e+1| - \ln|1| = \ln(e+1).$$

3.

Let $u = x$, $dv = \cos \pi x\, dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{\pi} \sin \pi x$. Then

$$\int_0^{1/2} x \cos \pi x\, dx = \left[\frac{1}{\pi} x \sin \pi x\right]_0^{1/2} - \int_0^{1/2} \frac{1}{\pi} \sin \pi x\, dx = \frac{1}{2\pi} - 0 - \frac{1}{\pi}\left[-\frac{1}{\pi} \cos \pi x\right]_0^{1/2}$$

$$= \frac{1}{2\pi} + \frac{1}{\pi^2}(0-1) = \frac{1}{2\pi} - \frac{1}{\pi^2} \text{ or } \frac{\pi - 2}{2\pi^2}$$

First let $u = x^2 + 1$, $dv = e^{-x}\, dx \quad \Rightarrow \quad du = 2x\, dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\, dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\, dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\, dx.$

Next let $U = x$, $dV = e^{-x}\, dx \quad \Rightarrow \quad dU = dx$, $V = -e^{-x}$. By (6) again,

$\int_0^1 xe^{-x}\, dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1.$ So

$\int_0^1 (x^2+1)e^{-x}\, dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3.$

Let $u = t$, $dv = \cosh t\, dt \quad \Rightarrow \quad du = dt$, $v = \sinh t$. Then

$\int_0^1 t \cosh t\, dt = \left[t \sinh t\right]_0^1 - \int_0^1 \sinh t\, dt = (\sinh 1 - \sinh 0) - \left[\cosh t\right]_0^1 = \sinh 1 - (\cosh 1 - \cosh 0)$

$= \sinh 1 - \cosh 1 + 1.$

We can use the definitions of sinh and cosh to write the answer in terms of e:

$\sinh 1 - \cosh 1 + 1 = \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^1 + e^{-1}) + 1 = -e^{-1} + 1 = 1 - 1/e.$

Let $u = \ln y$, $dv = \dfrac{1}{\sqrt{y}}\, dy = y^{-1/2}\, dy \quad \Rightarrow \quad du = \dfrac{1}{y}\, dy$, $v = 2y^{1/2}$. Then

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\, dy = \left[2\sqrt{y}\,\ln y\right]_4^9 - \int_4^9 2y^{-1/2}\, dy = (6\ln 9 - 4\ln 4) - \left[4\sqrt{y}\right]_4^9 = 6\ln 9 - 4\ln 4 - (12 - 8)$$

$$= 6\ln 9 - 4\ln 4 - 4$$

Let $u = \ln r$, $dv = r^3 dr \quad \Rightarrow \quad du = \frac{1}{r}\, dr$, $v = \frac{1}{4}r^4$. Then

$\int_1^3 r^3 \ln r\, dr = \left[\frac{1}{4}r^4 \ln r\right]_1^3 - \int_1^3 \frac{1}{4}r^3 dr = \frac{81}{4}\ln 3 - 0 - \frac{1}{4}\left[\frac{1}{4}r^4\right]_1^3 = \frac{81}{4}\ln 3 - \frac{1}{16}(81 - 1) = \frac{81}{4}\ln 3 - 5.$

First let $u = t^2$, $dv = \sin 2t\, dt \quad \Rightarrow \quad du = 2t\, dt$, $v = -\frac{1}{2}\cos 2t$. By (6),

$\int_0^{2\pi} t^2 \sin 2t\, dt = \left[-\frac{1}{2}t^2 \cos 2t\right]_0^{2\pi} + \int_0^{2\pi} t \cos 2t\, dt = -2\pi^2 + \int_0^{2\pi} t\cos 2t\, dt$. Next let $U = t$, $dV = \cos 2t\, dt \quad \Rightarrow$

$dU = dt$, $V = \frac{1}{2}\sin 2t$. By (6) again,

$\int_0^{2\pi} t\cos 2t\, dt = \left[\frac{1}{2}t \sin 2t\right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{2}\sin 2t\, dt = 0 - \left[-\frac{1}{4}\cos 2t\right]_0^{2\pi} = \frac{1}{4} - \frac{1}{4} = 0.$ Thus, $\int_0^{2\pi} t^2 \sin 2t\, dt = -2\pi^2.$

4. The integral represents the sum of the area of a triangle with height 1 and width 1 and of a semi-circle with radius 1, which is equal to

$\frac{1}{2} + \pi/4$