Calculus I: Homework 11 solutions
Ziad Saade

Problem 1: Solve the following integrals

$$\int \frac{12x}{\sqrt{2x^2+5}}\,dx$$

$$\int 2x\sin(3x)\,dx$$

$$\int_2^3 \frac{3x^2+2x+1}{x}\,dx = \left(\frac{3\cdot 3^2}{2}+2\cdot 3+\ln 3\right)-\left(\frac{3\cdot 2^2}{2}+2\cdot 2+\ln 2\right)$$

$$\int \tan x\,dx$$

= -log(cos(x)) + C

$$\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$$

Let $\left\{\begin{array}{l} u=\sqrt{x} \\ du=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}dx \end{array}\right\} \Rightarrow \quad \displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}dx = \int \arctan u \cdot 2du$

$= 2\displaystyle\int \arctan u \, du.$

By parts: $\left\{\begin{array}{ll} f=\arctan u & f'=\frac{1}{1+u^2} \\ g'=1 & g=u \end{array}\right\} \Rightarrow \quad = 2\left[u\arctan u - \displaystyle\int \frac{u}{1+u^2}du\right]$

$= 2u\arctan u - 2\displaystyle\int \frac{u}{1+u^2}du$

Let $\left\{\begin{array}{l} w=1+u^2 \\ dw=2u\,du \end{array}\right\} \Rightarrow \quad = 2u\arctan u - \displaystyle\int \frac{dw}{w}$

$= 2u\arctan u - \ln|w| + C$

$= 2u\arctan u - \ln|1+u^2| + C$

$= 2\sqrt{x}\arctan\sqrt{x} - \ln|1+(\sqrt{x})^2| + C$

$= \boxed{2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C}$

(Note: okay to drop absolute value bars, since it's understood here that $x>0$ (otherwise $\sqrt{x}$ would not be defined).)

$$\int_{-2}^{1} |x| \, dx.$$

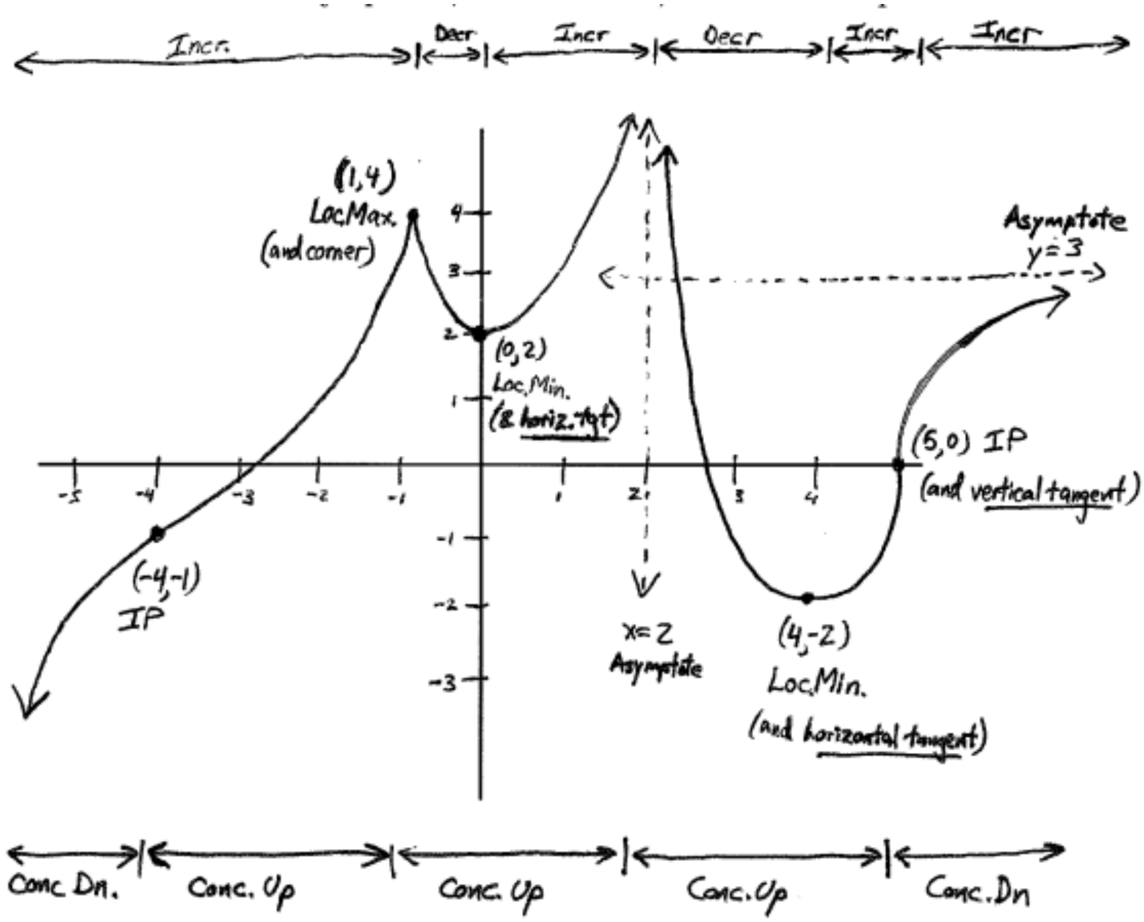
$$\int x^2 \log(x)\,dx$$

$$\int \frac{3\cos(\log(x))}{x}dx$$

Problem 2: This is more a review problem for the entire semester. Please explain briefly your sketch. A sketch alone will not receive full credits.

Sketch the graph of the function f that has the following properties: Please label all asymptotes

- $f(x)$ is continuous on its entire domain, which is all x except $x = 2$.
- $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = 3$.
- $\lim_{x\to 2} f(x) = \infty$.
- $f'(x)$ is continuous at all x except $x = -1$, $x = 2$, and $x = 5$.
- $f'(x) > 0$ for $x < -1$ and for $0 < x < 2$ and for $4 < x < 5$ and for $x > 5$.
- $f'(x) < 0$ for $-1 < x < 0$ and for $2 < x < 4$.
- $\lim_{x\to-1^-} f'(x) = 3$ and $\lim_{x\to-1^+} f'(x) = -3$.
- $\lim_{x\to 5} f'(x) = \infty$.
- $f''(x) > 0$ for $-4 < x < -1$ and for $-1 < x < 2$ and for $2 < x < 5$.
- $f''(x) < 0$ for $x < -4$ and for $x > 5$.
- $f(-4) = -1$, $f(-1) = 4$, $f(0) = 2$, $f(4) = -2$, and $f(5) = 0$.



Problem 3: Let's find other ways of computing area

1- Is the following integral equality true or false?

$$\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

TRUE:

$$\frac{d}{dx}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C\right) = \frac{1}{2}\sqrt{1-x^2} + \frac{x}{2}\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) + \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}\sqrt{1-x^2} + \frac{-x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot\left((1-x^2)+(-x^2)+1\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot(2-2x^2)$$

$$= \frac{1}{2}\cdot(1-x^2)^{-1/2}\cdot 2\cdot(1-x^2) = \boxed{\sqrt{1-x^2}}$$ as desired.

2- **Use the equality above** to compute the area of a circle of radius 1.

A semicircle of radius 1 centered at the origin has boundary curve given by $y=\sqrt{1-x^2}$ (from $x^2+y^2=1$).

Thus, area $= \displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx = \left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x\right]_{x=-1}^{x=1}$

$$=\left(\frac{1}{2}\cdot\sqrt{1-1}+\frac{1}{2}\arcsin 1\right) - \left(-\frac{1}{2}\sqrt{1-1}+\frac{1}{2}\arcsin(-1)\right)$$

$$= \frac{1}{2}\arcsin 1 - \frac{1}{2}\arcsin(-1) = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\left(-\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{2}}.$$

$\left(\sin\frac{\pi}{2}=1,\text{ so } \arcsin 1 = \frac{\pi}{2}.\right)$

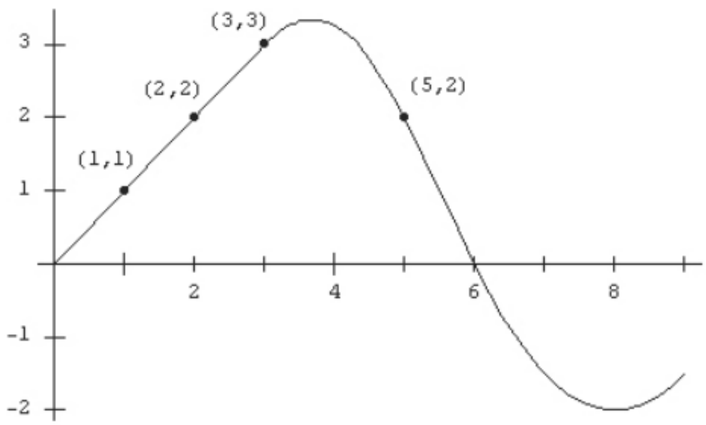
Problem 4: This problem is a good application of calculus in physics.

Consider the following graph.

This is the graph of the velocity of a particle that we denote as v(t) in meters per second.
Let s(t) (graph not shown) be the function of the **position** of the particle.

We have s(0) = 1 m

Please answer all questions with a clear explanation.
BACKGROUND INFO: The velocity function is the derivative of the position and the acceleration function is the derivative of the velocity vector



a) What is the particle's velocity at t= 5?

By the graph above, $v(5) = 2$ m/s.

b) Is the acceleration of the particle at time t=5 positive or negative?

Acceleration $a(t) = v'(t)$, so acceleration is given by the slope of the above curve. Thus, at $t=5$, acceleration is negative (v is decreasing at 5).

c) What is the particle's position at t=3?

By the net change theorem, $\begin{pmatrix}\text{change in position}\\ \text{between } t=0 \text{ \& } t=3\end{pmatrix} = \int_0^3 v(t)\,dt$

$= \begin{matrix}\text{Area under graph above}\\ \text{between } t=0 \text{ \& } t=3\end{matrix} = \frac{1}{2}(3)(3) = \frac{9}{2}$,

so position at $t=3$ is given by $s(3) = s(0) + (\text{change in pos})$

$= 1 + \frac{9}{2} = \boxed{\frac{11}{2}\ m}$.

d) At what time during the first 9 seconds does the position s have its largest value

Using same reasoning as in (c), we have $s(T) = s(0) + \int_0^T v(t)\,dt$.

So s is largest when $\int_0^T v(t)\,dt$ is largest, i.e. when area under above graph betw. 0 & T largest. This occurs at $\boxed{T=6}$ since for $T>6$ the area is reduced by the amount under the axis.

Approximately when is the acceleration zero?

e) Approximately when is the acceleration zero?

Since $a(t) = v'(t)$, we cite the values of t where graph above is horizontal: approx $t = 3\frac{3}{4}$ and $t = 8$ sec.

f) When is the particle moving **toward** the origin? **Away** from the origin?

g) On which side (positive or negative) of the origin does the particle lie at t= 9?

h) The integral of v(t) from 0 to 6 is 11.5
The integral of v(t) from 6 to 9 is -4.5
Find the total distance traveled by the particle the first 9 seconds.

Position starts as positive at $t=0$, and between $t=0$ & $t=6$ the velocity is positive, so position increases, meaning motion is away from origin.

After $t=6$, velocity is negative, meaning motion is toward origin (since position is positive, and according to part g, position never is negative).

(g) On which side (positive or negative) of the origin does the particle lie at time $t = 9$?

Position is positive: since $s(T) = s(0) + \int_0^T v(t)dt = 1 + \int_0^T v(t)dt$, position at time $t=T$ can only be negative if $\int_0^T v(t)dt$ is less than -1 for some T. But as noted, this area starts positive for small T, and although is reduced after $T=6$, the total area between $T=6$ and $T=9$ does not sufficiently reduce the positive area between 0 and 6.

(h) Given that $\int_0^6 v(t)\,dt = 11.5$ and $\int_6^9 v(t)\,dt = -4.5$, find the *total distance traveled* by the particle in the first 9 seconds.

$$\text{Total distance traveled} = \int_0^9 |v(t)|dt$$

$$= \int_0^6 v(t)dt + \left|\int_6^9 v(t)dt\right| \quad \left(\begin{array}{l}\text{because } v \geq 0 \text{ on } [0,6],\\ \text{and } v \leq 0 \text{ on } [6,9]\end{array}\right)$$

$$= 11.5 + 4.5 = \boxed{16\,m}.$$

Problem 5: State whether the following statements are True or False. If you think a statement is false, give a counter example. If it is true, you can just state it is. If you write a formal proof, you will get extra credit!

a) If f is continuous on $[a, b]$, then

$$\sqrt{\int_a^b f(x)\,dx} = \int_a^b \sqrt{f(x)}\,dx$$

FALSE: try f(x) =1

b) If f and g are continuous on $[a, b]$, then

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

TRUE

c) [3 points] $\int_{-1}^{3} 2x - x^2\,dx$ represents the area between the curve

$$y = 2x - x^2$$

the x-axis, $x = -1$ and $x = 3$.

False: it's the NET Area

d) If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x).$$

FALSE: $\frac{d}{dx}\left(\int_a^b f(x)dx\right) = 0$