Calculus 1 – Spring 2019 Section 2
HW12 (*Bonus*) *Solutions*

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Remark. There is no submission for this homework as it is the last one. Be sure to do it only after you: (1) finish reviewing HW1-HW11 (2) finish reviewing all midterms and practice midterms (3) finish reviewing the practice final.

1 Set theory

1.1 Exercise. (Power sets) Recall $\mathcal{P}(A)$ is *the power set of* A, i.e. the set of all subsets of A (see HW1). *Show* that for any two sets A and B,

$$A = B \quad \Longleftrightarrow \quad \mathcal{P}(A) = \mathcal{P}(B) \ .$$

Solution. In order to show $\Longleftrightarrow$, we must show both $\Longrightarrow$ and $\Longleftarrow$.

Let us start with $\Longrightarrow$. So assume that $A = B$. In that case, we would like to show that $\mathcal{P}(A) = \mathcal{P}(B)$. This statement itself entails two statements, namely, that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and $\mathcal{P}(A) \supseteq \mathcal{P}(B)$. Let us start with $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Take any $X \in \mathcal{P}(A)$. Then that means $X \subseteq A$. But $A = B$, so $X \subseteq B$ as well, i.e., $X \in \mathcal{P}(B)$. Since X was arbitrary, this now implies $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. The proof that $\mathcal{P}(A) \supseteq \mathcal{P}(B)$ holds goes along similar lines. We conclude $A = B \Longrightarrow \mathcal{P}(A) = \mathcal{P}(B)$.

We proceed with $\Longleftarrow$. So we assume that $\mathcal{P}(A) = \mathcal{P}(B)$ and we want to conclude that $A = B$. Again, $A = B$ entails that both $A \subseteq B$ and $A \supseteq B$ hold true. Let us start with $A \subseteq B$. To that end, pick any $x \in A$. We want to show that $x \in B$. Note that $\{ x \} \subseteq A$ (we just formed the singleton set). This means that $\{ x \} \in \mathcal{P}(A)$. But $\mathcal{P}(A) = \mathcal{P}(B)$, so $\{ x \} \in \mathcal{P}(B)$. But that means that $x \in B$. We conclude that $A \subseteq B$. Similarly one would conclude that $A \supseteq B$. So finally that means $A = B \Longleftarrow \mathcal{P}(A) = \mathcal{P}(B)$ and we're done.

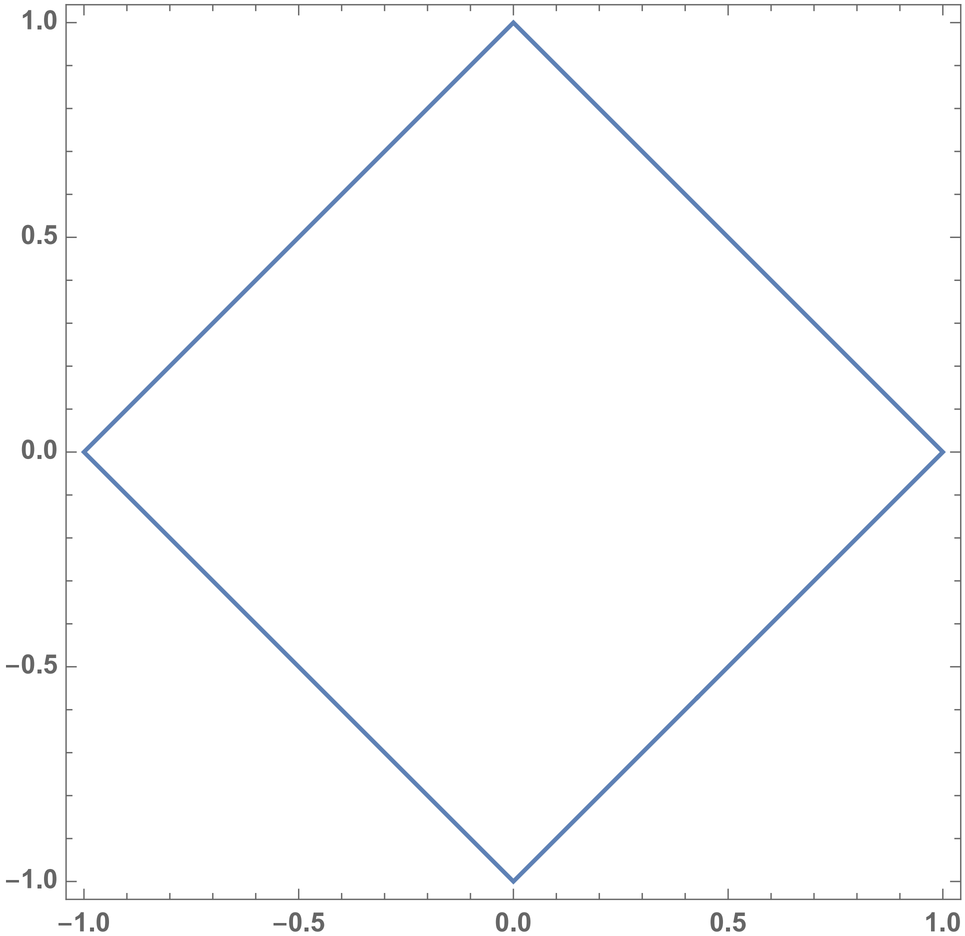
1.2 Exercise. (Set-builder notation) Sketch the following subset of $\mathbb{R}^2$:

$$\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, |x| + |y| = 1 \right\} \ .$$

Solution. The equation $|x| + |y| = 1$ can be simplified by dividing into four possible quadrants:

1. The first quadrant: $x \geq 0$ and $y \geq 0$. In this case, $|x| = x$ and $|y| = y$ and the constraint becomes $x + y = 1$ or $y = 1 - x$.
2. The second quadrant: $x < 0$ and $y \geq 0$. In this case, $|x| = -x$ and $|y| = y$ so we get $y = 1 + x$.
3. The third quadrant: $x < 0$ and $y < 0$, so $|x| = -x$ and $|y| = -y$ and so $y = -x - 1$.
4. The fourth quadrant: $x \geq 0$ and $y < 0$, so $|x| = x$ and $|y| = -y$. Hence $y = x - 1$.

All of these equations are straight lines. Note that when $x = 0$ then $y = \pm 1$ or when $x = \pm 1$ and $y = 0$, the constraint is fulfilled. Hence we now just have to plot four straight lines (each restricted to their respective quadrant). The result is:



1.3 Exercise. (Sets of numbers) Define a function $f:\mathbb{N}\to\mathbb{Z}^2$ such that (and *demonstrate* these facts):

1. It is well-defined (i.e. $f(n)$ really lands in $\mathbb{Z}^2$ for all $n\in\mathbb{N}$ and the definition specifies $f(n)$ as a unique element of $\mathbb{Z}^2$).

2. It is injective.

3. It is surjective.

Solution. We define $g:\mathbb{N}\to\mathbb{Z}$ by

$$g(n) \quad : \quad = \begin{cases} \frac{n-1}{2} & n\in 2\mathbb{N}+1 \\ -\frac{n}{2} & n\in 2\mathbb{N} \end{cases} .$$

Despite the division by 2, we really do land in $\mathbb{Z}$ and not in $\mathbb{Q}$: When n is odd, we first subtract one (so $n-1$ is even) so that dividing by 2 still results in an integer. When n is even, dividing by two still results in an integer. Hence the domain of g being $\mathbb{Z}$ makes sense. Also note that since we are given an explicit formula, there is exactly one unique element of $\mathbb{Z}$ specified for each $n\in\mathbb{N}$. We conclude g is well-defined.

Next, g is injective: If $g(n)=g(m)$ for $n,m\in\mathbb{N}$, then

1. If both n,m are even: $-\frac{n}{2}=-\frac{m}{2}$ which implies $n=m$.

2. If both n,m are odd: $\frac{n-1}{2}=\frac{m-1}{2}$, which implies $n=m$.

3. If n is even and m is odd, $-\frac{n}{2}=\frac{m-1}{2}$ which implies $-n=m-1$, or $n+m=1$ which is impossible because both $n,m\geq 1$, so $n+m$ is at least 2.

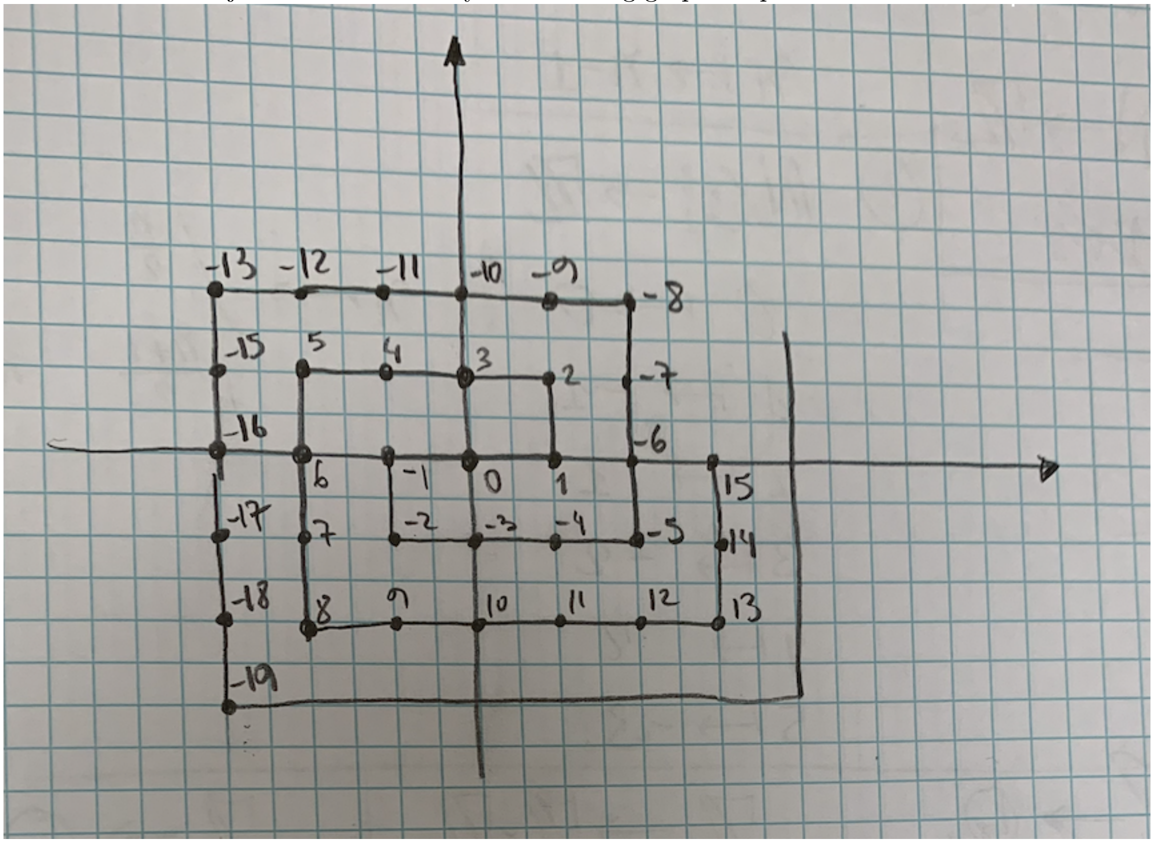
4. Similarly it is impossible that n is odd and m is even.

We conclude that g is indeed injective.

g is also surjective: Let $a\in\mathbb{Z}$ be given. Then if $a\geq 0$, $f(2a+1)=a$. If $a<0$, $f(-2a)=a$.

I.e., g is bijective (=injective and surjective).

Next we define a bijection $h : \mathbb{Z}^2 \to \mathbb{Z}$ by the following graphical procedure:



You can convince yourself that this map is really well-defined, and bijective. Hence $f = g \circ h^{-1}$ is the sought after map.

2 Functions

2.1 Exercise. Give an example of a function $f : \{ 1, 2, 3 \} \to \{ 1, 2, 3, 4 \}$ which is not injective.

Solution. Consider the function $f(n) := 1$ for all $n \in \{ 1, 2, 3 \}$. Since 1 is covered three times, the function is *not* injective.

2.2 Exercise. What is the subset of $\{ 1, 2, 3 \} \times \{ 1, 2, 3, 4 \}$ which corresponds to the function $f : \{ 1, 2, 3 \} \to \{ 1, 2, 3, 4 \}$ given by

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 3 \\ 3 &\mapsto 4\,? \end{aligned}$$

It might be helpful to draw a table.

Solution. We make the table which corresponds to the product set $\{ 1, 2, 3 \} \times \{ 1, 2, 3, 4 \}$:

	1	2	3	4
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$

Now we color all those pairs that correspond to the function:

	1	2	3	4
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$

I.e., the function f may be associated with the subset of $\{ 1, 2, 3 \} \times \{ 1, 2, 3, 4 \}$ given by

$$\{ (1, 2), (2, 3), (3, 4) \} .$$

2.3 Exercise. Give a definition of cos, sin, exp and log: For each of them:

1. Describe what they do (you probably can't write down a formula), i.e., how are they defined? The answer to which question do they provide?
2. Describe the largest possible domain one could pick.
3. Describe the smallest possible codomain one could pick for that choice of domain from the step above (i.e. their image, or range).
4. Is the function bounded?
5. What value does it approach (if any) on the left and right of its largest domain?
6. For which values of its argument does it yield the value zero?
7. What is its value when the argument is zero?
8. Where are they continuous?
9. Where are they differentiable?
10. Where are they increasing / decreasing?

Finally, make a sketch.

Solution. The solution is omitted since it appears in Section 10 of the lecture notes.

3 Limits

3.1 Exercise. Give an example for a sequence $a : \mathbb{N} \to \mathbb{R}$ which is: (1) monotone increasing (2) bounded from above (3) *not* convergent.

Solution. This is impossible, as stipulated by Claim 6.17 in the lecture notes.

3.2 Exercise. Using the formula $\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$, which is valid for any $n \in \mathbb{N}$, $a \in \mathbb{R}$ and $r \neq 1$, evaluate the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} r^k \quad = \quad ?$$

when $r < 1$. In each of your steps, indicate which property of limits you are using in order to proceed.

Solution. Using the formula with $a = 1$ we get

$$\sum_{k=0}^{n-1} r^k \quad = \quad \frac{1-r^n}{1-r}$$

Since the only dependence on n is in the numerator, we have

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=0}^{n-1} r^k &= \lim_{n\to\infty} \frac{1-r^n}{1-r} \\ &\quad \text{(Algebra of limits)} \\ &= \underbrace{\left(\lim_{n\to\infty} \frac{1}{1-r}\right)}_{\text{sequence constant in n, so the result of the limit is that constant, } \frac{1}{1-r}} \left(\lim_{n\to\infty} 1 - r^n\right) \\ &= \frac{1}{1-r}\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} r^n\right) . \end{aligned}$$

Now by Claim 6.16 item (5) we conclude that since $r < 1$, $\lim_{n\to\infty} r^n = 0$. The other remaining limit is the constant 1 sequence, and hence, we find the final result

$$\frac{1}{1-r} .$$

3.3 Exercise. Consider the function $f:\mathbb{R}\to\mathbb{R}$ given by the piecewise formula

$$\mathbb{R}\ni x \quad\mapsto\quad \begin{cases}\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}\in\mathbb{R}\,.$$

1. Evaluate

$$\lim_{x\to\infty} f(x) \quad=\quad ?$$

2. Evaluate

$$\lim_{x\to-\infty} f(x) \quad=\quad ?$$

3. Evaluate

$$\lim_{x\to 0} f(x) \quad=\quad ?$$

Does it even exist? How about

$$\lim_{x\to 0} xf(x) \quad=\quad ?$$

and

$$\lim_{x\to\infty} xf(x) \quad=\quad ?$$

Solution. The first limit equals zero. This is because sin is continuous at zero and equals zero there, and $\frac{1}{x}\to 0$ as $x\to\infty$.

The same for the second limit, again, since $\frac{1}{x}\to 0$ as $x\to-\infty$.

The third limit however does *not* exist. Indeed, intuitively, we are evaluating sin as its argument approaches ∞, and this just keeps oscillating. More formally, whatever candidate value $L\in[-1,1]$ one could propose, for any $\varepsilon>0$, if we want to arrange that $\left|\sin\left(\frac{1}{x}\right)-L\right|<\varepsilon$, we'll need

$$L-\varepsilon<\quad\sin\left(\tfrac{1}{x}\right)\quad<L+\varepsilon\,.$$

However, as x gets smaller and smaller, sin will necessarily span *all* values between $[-1,1]$ repeatedly so that really it cannot be constrained to with $(L-\varepsilon,L+\varepsilon)$. Hence this limit cannot exist.

For $xf(x)$ as $x\to 0$, we can use the squeeze theorem with $\operatorname{im}(\sin)\subseteq[-1,1]$ to get

$$-x\leq\quad x\sin\left(\tfrac{1}{x}\right)\quad\leq x$$

so that $\lim_{x\to 0} xf(x)=0$ as both sides go to zero.

The limit $xf(x)$ as $x\to\infty$ is *not* convergent, because it keeps oscillating (as before) between $\pm\infty$ (informally speaking).

4 Continuity

4.1 Exercise. Is the function f from the previous exercise continuous? If not, where does it fail to be continuous?

Solution. That function is *not* continuous at zero (otherwise it is continuous everywhere). Indeed, for it to be continuous, we would need to have

$$\begin{aligned}\lim_{x\to 0}\sin\left(\frac{1}{x}\right) &= f(0)\\ &\equiv 0\,.\end{aligned}$$

However, that limit does not even exist. Since both $\mathbb{R}\setminus\{0\}\ni x\mapsto\frac{1}{x}$ and sin are continuous, the composition is, and so f is continuous on $\mathbb{R}\setminus\{0\}$.

4.2 Exercise. Give an example of a function f which is discontinuous and another function g, such that $g\circ f$ is none the less continuous.

Solution. Take f from above and $g=1$, the constant 1 function. Then $g\circ f$ is the constant 1 function, which is continuous everywhere.

4.3 Exercise. If $f:[-1,1]\to\mathbb{R}$ is continuous, and $f(-1)=10$ and $f(1)=-10$, is there a solution for $x\in\mathbb{R}$ to the equation

$$f(x)\quad=\quad 0\,?$$

Solution. Using the intermediate value theorem, since f is continuous, there must be some point $t\in(-1,1)$ such that $f(t)=0$. That t is the solution to the above equation.

5 Differentiation

5.1 Exercise. Recall that one could define $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, and that with this restriction of domain, tan is a bijective function. Hence there is an inverse, which is called *the arctangent*, $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means that (with the notation for *the identity function* of a set A given by $\mathbb{1}_A(a) \equiv a$ for any $a \in A$)

$$\begin{aligned} \tan \circ \arctan &= \mathbb{1}_{\mathbb{R}} \\ \arctan \circ \tan &= \mathbb{1}_{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} \,. \end{aligned}$$

Differentiate both sides of one of the above equations using the chain rule, re-arrange, and use trigonometric identities, in order to find a formula for $\arctan'$ (the derivative of arctan) using $\tan'$ (which you should know) and arctan itself.

Solution. We have (an equation for functions, where on the RHS the function is the constant 1 function)

$$(\arctan' \circ \tan) \tan' = 1$$

Furthermore, we know that $\tan' = \frac{1}{\cos^2}$, so we find

$$\arctan' \circ \tan = \frac{1}{\cos^2}$$

or

$$\arctan' = \frac{1}{(\cos \circ \arctan)^2} \,.$$

To finish we should understand what $\cos \circ \arctan$ is. It would help to re-write cos in terms of tan. For example:

$$\tan \equiv \frac{\sin}{\cos}$$

and the Pythagorian theorem implies

$$\sin = \sqrt{1 - \cos^2} \,.$$

So

$$\tan = \frac{\sqrt{1 - \cos^2}}{\cos} \,.$$

Solving for cos we find

$$\cos = \frac{1}{\sqrt{\tan^2 + 1}}$$

and so using $\tan \circ \arctan = \mathbb{1}$,

$$(\cos \circ \arctan)(x) = \frac{1}{\sqrt{x^2 + 1}} \,.$$

We learn that

$$\arctan'(x) = \frac{1}{x^2 + 1} \,.$$

5.2 Exercise. Is $\sqrt{\cdot} : [0, \mathbb{R}) \to \mathbb{R}$ continuous? Is it differentiable? Where does it fail to be differentiable? Show why.

Solution. $\sqrt{\cdot}$ is continuous on $[0, \mathbb{R})$. Indeed, if $y \geq 0$, we'd want

$$\lim_{x \to y} \sqrt{x} = \sqrt{y} \,.$$

Consider the factorization in Claim 11.2 in the lecture notes, which gives us:

$$\begin{aligned} \left|\sqrt{x} - \sqrt{y}\right| &= \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \\ &= \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \,. \end{aligned}$$

Now if $y > 0$, then we have

$$\sqrt{x} + \sqrt{y} \quad \geq \quad \sqrt{y}$$

and so

$$\left|\sqrt{x} - \sqrt{y}\right| \quad \leq \quad \frac{|x-y|}{\sqrt{y}}.$$

So if $x > 0$ is chosen such that $0 < |x-y| \leq \sqrt{y}\varepsilon$ then $\left|\sqrt{x} - \sqrt{y}\right| \leq \varepsilon$, and this is true for any $\varepsilon > 0$. Hence the limit exists, and converges to $\sqrt{y}$. Conversely, if $y = 0$, we have if x is chosen such that $0 < x < \varepsilon^2$, then $\sqrt{x} < \varepsilon$ and so the limit also exists and equals zero as desired. Note how we had to separate into two cases.

For differentiability at $y \geq 0$, we need to study the limit

$$\sqrt{y}' \quad \overset{?}{=} \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\sqrt{y+\varepsilon} - \sqrt{y}\right) .$$

Again using $\sqrt{y+\varepsilon} - \sqrt{y} = \frac{y+\varepsilon-y}{\sqrt{y+\varepsilon}+\sqrt{y}} = \frac{\varepsilon}{\sqrt{y+\varepsilon}+\sqrt{y}}$ we learn that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\sqrt{y+\varepsilon} - \sqrt{y}\right) \quad \overset{?}{=} \quad \lim_{\varepsilon \to 0} \frac{1}{\sqrt{y+\varepsilon} + \sqrt{y}} .$$

Now we separate our study into two cases. If $y > 0$, then using the continuity of $\sqrt{\cdot}$ we just proved we find

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\sqrt{y+\varepsilon} + \sqrt{y}} \quad &= \quad \frac{1}{\lim_{\varepsilon \to 0}\left(\sqrt{y+\varepsilon} + \sqrt{y}\right)} \\
&= \quad \frac{1}{\lim_{\varepsilon \to 0}\sqrt{y+\varepsilon} + \lim_{\varepsilon \to 0}\sqrt{y}} \\
&\quad\quad \left(\text{Use continuity of } \sqrt{\cdot}\right) \\
&= \quad \frac{1}{\sqrt{\lim_{\varepsilon \to 0} y+\varepsilon} + \sqrt{y}} \\
&= \quad \frac{1}{2\sqrt{y}} .
\end{aligned}$$

On the other hand, if $y = 0$, we have to evaluate$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}}$. This last limit, however, diverges to infinity! Hence $\sqrt{\cdot}$ is not differentiable at zero.

5.3 Exercise. Use the mean value theorem in order to prove the estimate

$$\log(x) \quad \leq \quad \frac{1}{s}(x^s - 1) \qquad (x > 0, s > 0) .$$

Hint: divide into two cases: when $x < 1$ and $x > 1$, and choose the interval on which you'll apply the MVT accordingly.

Use this estimate and the logarithm laws in order to show that

$$\log(x) \quad \geq \quad s\left(1 - \frac{1}{x^s}\right) \qquad (x > 0, s > 0) .$$

Solution. We start with the more basic inequality

$$\log(x) \quad \leq \quad x - 1 \qquad (x > 0) .$$

Divide into two cases: $x > 1$ and $x < 1$. (If $x = 1$, we get $\log(1) = 0$ and $x - 1 = 0$, so the inequality is an equality). So if $x > 1$, let us apply the mean value theorem on log (which is differentiable) between $(1, x)$ to get that there must be some point $t \in (1, x)$ such that

$$\log'(t) \quad = \quad \frac{\log(x) - \log(1)}{x - 1} .$$

Now, $\log'(t) = \frac{1}{t}$ and $t > 1$, so $\frac{1}{t} < 1$. Also we use $\log(1) = 0$. So

$$\begin{aligned}
\frac{\log(x)}{x-1} \quad &= \quad \frac{1}{t} \\
&< \quad 1
\end{aligned}$$

which is equivalent to

$$\log(x) \quad < \quad x-1\,.$$

Let now $x\in(0,1)$ be given. Then we may apply the MVT on log on the interval $(x,1)$ to get some $t\in(x,1)$ such that

$$\begin{aligned}\frac{1}{t} &= \frac{\log(1)-\log(x)}{1-x}\\ &= \frac{-\log(x)}{1-x}\end{aligned}$$

Now since $t<1$, $\frac{1}{t}>1$ and we learn that

$$\frac{-\log(x)}{1-x} \quad > \quad 1\,,$$

i.e.,

$$\log(x) \quad < \quad x-1$$

as desired.

Now let $s>0$. Then

$$\log(x^s) \quad = \quad s\log(x)$$

but also

$$\log(x^s) \quad \leq \quad x^s-1$$

so

$$\begin{aligned}\log(x) &= \frac{1}{s}(\log(x^s))\\ &\leq \frac{1}{s}(x^s-1)\ .\end{aligned}$$

Furthermore,

$$\log\left(x^{-s}\right) \quad = \quad -s\log(x)$$

but also

$$\log\left(x^{-s}\right) \quad \leq \quad x^{-s}-1$$

so

$$\begin{aligned}\log(x) &= -\frac{1}{s}\log\left(x^{-s}\right)\\ &\geq -\frac{1}{s}\left(x^{-s}-1\right)\\ &= \frac{1}{s}\left(1-x^{-s}\right)\ .\end{aligned}$$

6 Integration

6.1 Exercise. The equation for some ellipse is given by

$$\frac{x^2}{4}+y^2 \quad = \quad 1$$

(i.e. the set of all points $(x,y)\in\mathbb{R}^2$ obeying the above equation). Use the equation

$$\int_0^x\sqrt{1-z^2}\mathrm{d}z \quad = \quad \frac{1}{2}\left(x\sqrt{1-x^2}+\arcsin(x)\right) \qquad (0<x<1)$$

as well as the change of variables formula in order to calculate the total area of the ellipse.

Hint: It might help to make a picture. In that picture, divide the area you want to calculate into four equal parts (by symmetry). Use the formula above to calculate only one fourth of it.

Solution. The equation for the ellipse implies $y=\sqrt{1-\frac{x^2}{4}}$. Note that x ranges between -2 and 2, and y ranges between -1 and 1 as we traverse the entire ellipse. Since the ellipse is symmetric about the origin, we could just integrate the function $f:[0,2]\to\mathbb{R}$ given by

$$f(x) = \sqrt{1-\frac{x^2}{4}}$$

and multiply by *four* in order to get the area of the ellipse. Hence

$$\begin{aligned} A &= 4\times\int_0^2 f \\ &= 4\times\int_0^2\sqrt{1-\frac{x^2}{4}}\mathrm{d}x \end{aligned}$$

Now we do a change of variable: $\varphi(x):=\frac{x}{2}$. So $\varphi'(x)=\frac{1}{2}$ and we find

$$\sqrt{1-\frac{x^2}{4}} = 2\sqrt{1-\varphi(x)^2}\varphi'(x)$$

and so

$$\begin{aligned} A &= 4\times\int_0^2 2\sqrt{1-\varphi(x)^2}\varphi'(x)\,\mathrm{d}x \\ &\quad \text{(Use change of var. theorem)} \\ &= 8\int_0^1\sqrt{1-x^2}\mathrm{d}x \\ &\quad \text{(Use given formula)} \\ &= 8\left(\frac{1}{2}(\arcsin(1))\right) \\ &= 2\pi\,. \end{aligned}$$

Note that the formula for the area of the ellipse is πab, where a and b are the two radii of the ellipse (here $a=1$ and $b=2$).

6.2 Exercise. Evaluate

$$\int_0^x\frac{1}{\sqrt{1+y^2}}\mathrm{d}y = ?$$

Hint: use the change of variable formula with arcsinh.

Solution. We have $\varphi(y):=\operatorname{arcsinh}(y)$, so (as it turns out... you may verify this by following Claim 8.34 and known derivatives of the hyperbolic trigonometric functions) $\varphi'(y)=\frac{1}{\sqrt{1+y^2}}$. Then with $f=1$ (the constant 1 function) we have

$$\frac{1}{\sqrt{1+y^2}} = (f\circ\varphi)(y)\varphi'(y)$$

so

$$\int_0^x\frac{1}{\sqrt{1+y^2}} = \int_{\varphi(0)}^{\varphi(x)}\mathrm{d}y$$

Recall that $\operatorname{arcsinh}(0)=0$ and so $\int_0^x\frac{1}{\sqrt{1+y^2}}=\operatorname{arcsinh}(x)$.

Note: alternatively one could just use the fundamental theorem of calculus directly without change of variables!

6.3 Exercise. A ball was dropped at rest from height $y=1$ (measured in meters) at time zero. Calculate how long it takes for the ball to hit the ground, assuming that it is subject to acceleration at rate 10 (measured in meters per second).

Solution. Note the following relationships (as functions) between acceleration and position:

$$a = h''$$

Here we have a a constant function equal to -10 (since acceleration is working downwards, and we measure height so that positive values are higher), so

$$h'(t) = -10t + C$$

for *some* constant C (verify this by differentiating) and so

$$h(t) = -\frac{1}{2} \times 10 \times t^2 + Ct + D$$

for some other constant D.

To find C, D, we need to use the given data. Via the word "at rest", we learn that the initial velocity of the ball was zero, i.e., $h'(0) = 0$. This implies that $C = 0$. Since the initial height is 1, we learn that $D = 1$. Hence we find

$$h(t) = -5t^2 + 1$$

Now we want to solve $h(t) = 0$ for t, from which we find

$$t = \frac{1}{\sqrt{5}}$$

(we discard the $t < 0$ solution!).