Calculus 1 – Spring 2019 Section 2
HW12 (*Bonus*)

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Remark. There is no submission for this homework as it is the last one. Be sure to do it only after you: (1) finish reviewing HW1-HW11 (2) finish reviewing all midterms and practice midterms (3) finish reviewing the practice final.

1 Set theory

1.1 Exercise. (Power sets) Recall $\mathcal{P}(A)$ is *the power set of* A, i.e. the set of all subsets of A (see HW1). *Show* that for any two sets A and B,

$$A = B \iff \mathcal{P}(A) = \mathcal{P}(B)\ .$$

1.2 Exercise. (Set-builder notation) Sketch the following subset of $\mathbb{R}^2$:

$$\{\, (x, y) \in \mathbb{R}^2 \mid |x| + |y| = 1 \,\}\ .$$

1.3 Exercise. (Sets of numbers) Define a function $f : \mathbb{N} \to \mathbb{Q}$ such that (and *demonstrate* these facts):

1. It is well-defined (i.e. $f(n)$ really lands in $\mathbb{Q}$ for all $n \in \mathbb{N}$ and the definition specifies $f(n)$ as a unique element of $\mathbb{Q}$).
2. It is injective.
3. It is surjective.

2 Functions

2.1 Exercise. Give an example of a function $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ which is not injective.

2.2 Exercise. What is the subset of $\{1, 2, 3\} \times \{1, 2, 3, 4\}$ which corresponds to the function $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ given by

$$\begin{array}{ccc} 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 4\,? \end{array}$$

It might be helpful to draw a table.

2.3 Exercise. Give a definition of cos, sin, exp and log: For each of them:

1. Describe what they do (you probably can't write down a formula), i.e., how are they defined? The answer to which question do they provide?
2. Describe the largest possible domain one could pick.
3. Describe the smallest possible codomain one could pick for that choice of domain from the step above (i.e. their image, or range).
4. Is the function bounded?
5. What value does it approach (if any) on the left and right of its largest domain?

6. For which values of its argument does it yield the value zero?
7. What is its value when the argument is zero?
8. Where are they continuous?
9. Where are they differentiable?
10. Where are they increasing / decreasing?

Finally, make a sketch.

3 Limits

3.1 Exercise. Give an example for a sequence $a : \mathbb{N} \to \mathbb{R}$ which is: (1) monotone increasing (2) bounded from above (3) *not* convergent.

3.2 Exercise. Using the formula $\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$, which is valid for any $n \in \mathbb{N}$, $a \in \mathbb{R}$ and $r \neq 1$, evaluate the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} r^k \quad = \quad ?$$

when $r < 1$. In each of your steps, indicate which property of limits you are using in order to proceed.

3.3 Exercise. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the piecewise formula

$$\mathbb{R} \ni x \quad \mapsto \quad \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \in \mathbb{R}\,.$$

1. Evaluate

$$\lim_{x\to\infty} f(x) \quad = \quad ?$$

2. Evaluate

$$\lim_{x\to-\infty} f(x) \quad = \quad ?$$

3. Evaluate

$$\lim_{x\to 0} f(x) \quad = \quad ?$$

Does it even exist? How about

$$\lim_{x\to 0} x f(x) \quad = \quad ?$$

and

$$\lim_{x\to\infty} x f(x) \quad = \quad ?$$

4 Continuity

4.1 Exercise. Is the function f from the previous exercise continuous? If not, where does it fail to be continuous?

4.2 Exercise. Give an example of a function f which is discontinuous and another function g, such that $g \circ f$ is none the less continuous.

4.3 Exercise. If $f : [-1, 1] \to \mathbb{R}$ is continuous, and $f(-1) = 10$ and $f(1) = -10$, is there a solution for $x \in \mathbb{R}$ to the equation

$$f(x) \quad = \quad 0\,?$$

5 Differentiation

5.1 Exercise. Recall that one could define $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, and that with this restriction of domain, tan is a bijective function. Hence there is an inverse, which is called *the arctangent*, $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means that (with the notation for *the identity function* of a set A given by $\mathbb{1}_A(a) \equiv a$ for any $a \in A$)

$$\begin{aligned} \tan \circ \arctan &= \mathbb{1}_{\mathbb{R}} \\ \arctan \circ \tan &= \mathbb{1}_{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} . \end{aligned}$$

Differentiate both sides of one of the above equations using the chain rule, re-arrange, and use trigonometric identities, in order to find a formula for $\arctan'$ (the derivative of arctan) using $\tan'$ (which you should know) and arctan itself.

5.2 Exercise. Is $\sqrt{\cdot} : [0, \mathbb{R}) \to \mathbb{R}$ continuous? Is it differentiable? Where does it fail to be differentiable? Show why.

5.3 Exercise. Use the mean value theorem in order to prove the estimate

$$\log(x) \quad \leq \quad \frac{1}{s}(x^s - 1) \qquad (x > 0, s > 0) .$$

Hint: divide into two cases: when $x < 1$ and $x > 1$, and choose the interval on which you'll apply the MVT accordingly.

Use this estimate and the logarithm laws in order to show that

$$\log(x) \quad \geq \quad s\left(1 - \frac{1}{x^s}\right) \qquad (x > 0, s > 0) .$$

6 Integration

6.1 Exercise. The equation for some ellipse is given by

$$\frac{x^2}{4} + y^2 \quad = \quad 1$$

(i.e. the set of all points $(x, y) \in \mathbb{R}^2$ obeying the above equation). Use the equation

$$\int_0^x \sqrt{1 - z^2}\mathrm{d}z \quad = \quad \frac{1}{2}\left(x\sqrt{1 - x^2} + \arcsin(x)\right) \qquad (0 < x < 1)$$

as well as the change of variables formula in order to calculate the total area of the ellipse.

Hint: It might help to make a picture. In that picture, divide the area you want to calculate into four equal parts (by symmetry). Use the formula above to calculate only one fourth of it.

6.2 Exercise. Evaluate

$$\int_0^x \frac{1}{\sqrt{1 + y^2}}\mathrm{d}y \quad = \quad ?$$

Hint: use the change of variable formula with arcsinh.

6.3 Exercise. A ball was dropped at rest from height $y = 1$ (measured in meters) at time zero. Calculate how long it takes for the ball to hit the ground, assuming that it is subject to acceleration at rate 10 (measured in meters per second).