Calculus 1 HW2 Solution

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Functions

Exercise 1

Suppose that there is a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly increasing but not injective. Because f is not injective, there must exist two numbers a, b in $\mathbb{R}$ such that $a \neq b$ but $f(a) = f(b)$. If we assume $a < b$ without loss of generality, $f(a) < f(b)$ must hold as f is strictly increasing, which is a contradiction to the identity $f(a) = f(b)$ above. Therefore, there is no such function.

Exercise 2

Let x, y, z be 3 different elements in the domain A.

1. First we assume that f is injective, and also assume $|B| < 3$.
Case 1 : If $|B| = 1$, then for all 3 elements x, y, z, the function values should be the same $f(x) = f(y) = f(z)$, as there is only one element in codomain. This violates the fact that f is injective.
Case 2 : If $|B| = 2$, similar situation happens; there must be at least two different elements in A which have the same function value. This is also a contradiction that f is injective.
Therefore, if f is injective, the number of elements of B should be greater than or equal to 3.

2. We also prove by contradiction; we assume that f is surjective, as well as $|B| > 4$. Then, the set $B - \{f(x), f(y), f(z)\}$ is not empty and α is an element in this set. Then, there is no element in A with the function value

equal to α, which means f cannot be surjective. Thus, if f is surjective, the inequality $|B| \leq 3$ must hold.

3. f is bijective when it is both injective and surjective. From part 1 and 2, two inequalities $|B| \geq 3$ and $|B| \leq 3$ must hold. Combining these two, we have $|B| = 3$.

Exercise 3

1. First, the domain of f is $[-1, \infty)$, and the domain of g is $\mathbb{R}$.
$(f \circ g)(x) = \sqrt{\sin(x) + 1}$ and the domain is $\mathbb{R}$ (The range of sin function is $[-1, 1]$, thus $(f \circ g)(x)$ is well-defined for all values $x \in \mathbb{R}$).
$(g \circ f)(x) = \sin(\sqrt{x+1})$, and the domain is $[-1, \infty)$ (same as f).

2. The domains of f and g are both $\mathbb{R}$.
$(f \circ g)(x) = e^{-x^2+4x}$, and $(g \circ f)(x) = -e^{2x} + 4e^{x}$ and the domains are also $\mathbb{R}$.

4. Let $f(x) = \sin^2(x)$ and $g(x) = \frac{1}{x}$. Then, $(f \circ g)(x) = \sin^2(\frac{1}{x})$.

5. Let $f(x) = \sqrt{x}$ and $g(x) = 2|x|$. Then, $(f \circ g)(x) = \sqrt{2|x|}$.

6. Let $f(x) = 5x + 6$ and $g(x) = \sin x$. Then, $(f \circ g)(x) = 5\sin(x) + 6$.

Exercise 4

Graphs are on the last page.

1. The range is $(-\infty, 4]$. $f(x) = -(x-3)^2 + 4$

2. The range is $[-4, 4]$. $f(x) = 4\sin(x - \frac{\pi}{2})$

3. The range is $(5, \infty)$. $f(x) = e^{-x} + 5$

4. $|x| + |y| = 1$

5. $xy = 0$

6. $x^2 = y^2$

Limits

Exercise 5

1. $a(n) = \frac{1}{2n}$ and the sequence converges to 0.

2. By the 5th statement in Claim 6.14 of Lecture note with $a = -\frac{1}{4}$, this sequence converges to 0.

3. This sequence does not converge because it oscillates between 3 numbers $1, 0, -1$.

4. $a(n) = \frac{n^2}{n+1}$ and the numerators grow much faster than the denominators. Thus, the limit diverges to ∞. (Actually, by setting the new sequence $b(n) = n^2 - 1$, one can show that $a(n) = \frac{n^2}{n+1} > n^2 - 1 = b(n)$. Try to show this! Then, it is also easy to show that $b(n)$ diverges to ∞, and the divergence of $a(n)$ follows.)

Exercise 6

1. $\lim\limits_{x\to 0} \frac{x^2-1}{x-1} = \lim\limits_{x\to 0} (x+1) = 1$

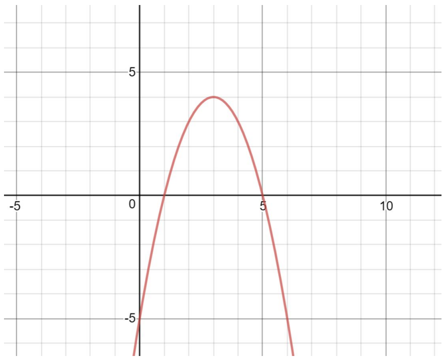
2. $\lim\limits_{x\to \infty} \frac{x^2-1}{x-1} = \lim\limits_{x\to \infty} (x+1) = \infty$ (Diverge)

3. $\lim\limits_{x\to 0} 2^{2^x} = 2^{\lim\limits_{x\to 0} 2^x} = 2^1 = 2$

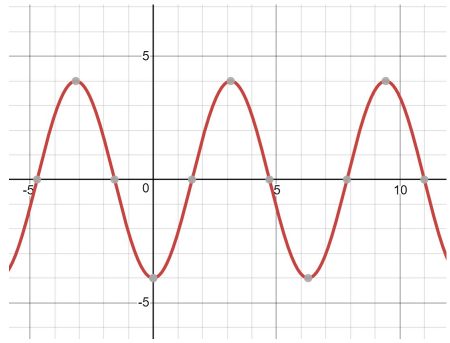
4. The inequality $-\frac{1}{x} \leq \frac{sin(5x)}{x} \leq \frac{1}{x}$ holds for positive x and we know that $\lim\limits_{x\to\infty} (-\frac{1}{x}) = \lim\limits_{x\to\infty} \frac{1}{x} = 0$. Thus, by the squeeze theorem, $\lim\limits_{x\to\infty} \frac{sin(5x)}{x} = 0$.

5. Because $\lim\limits_{x\to 0} \sin(\frac{1}{x})$ does not exist (it oscillates between the values in $[-1, 1]$), $\lim\limits_{x\to 0} \cos(3x) \cdot \sin(\frac{1}{x})$ also does not exist, even though we have $\lim\limits_{x\to 0} \cos(3x) = 1$.

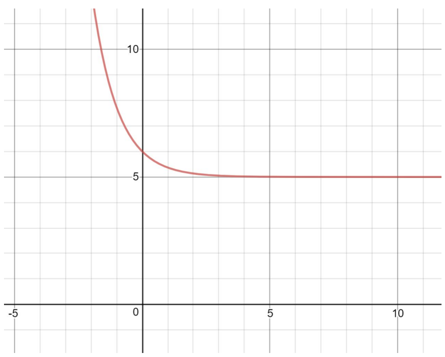
Figure 1: Exercise 4 Graphs



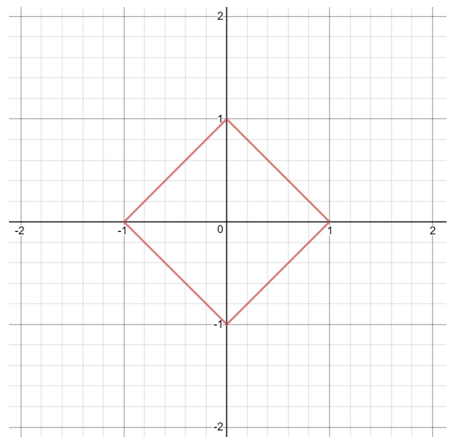
(a) Problem 1
$f(x) = -(x-3)^2 + 4$



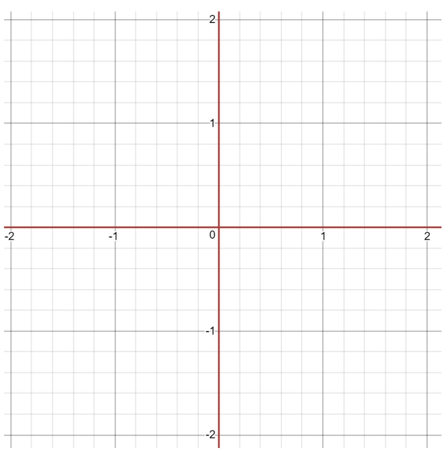
(b) Problem 2
$f(x) = 4\sin(x - \frac{\pi}{2})$



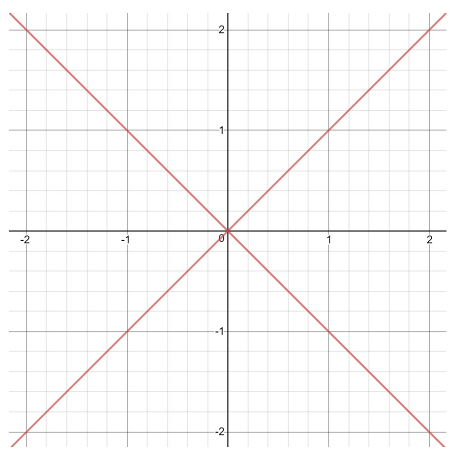
(c) Problem 3
$f(x) = e^{-x} + 5$



(d) Problem 4
$|x| + |y| = 1$



(e) Problem 5
$xy = 0$



(f) Problem 6
$x^2 = y^2$