Calculus 1–Section 2–Spring 2019–HW3 Solutions

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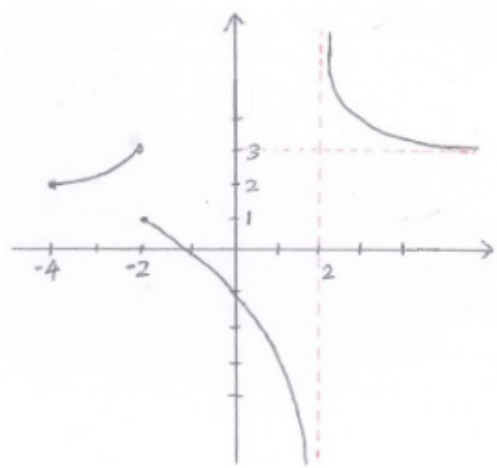
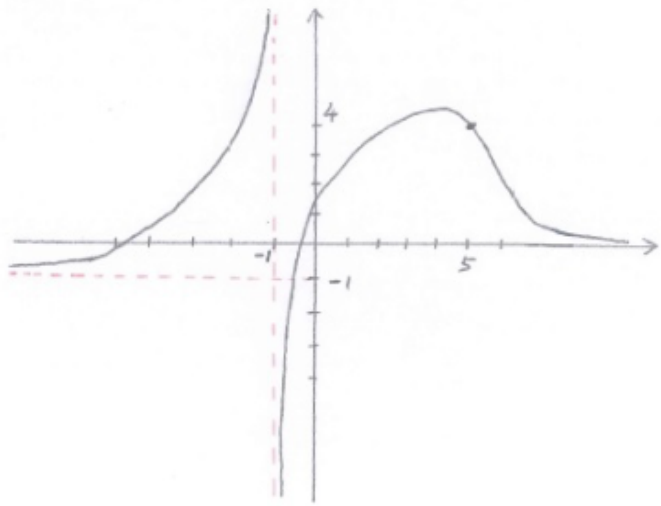
Exercise 1





Exercise 2

(1) For $f(x) = \log(\tan^3(2x))$ to be well-defined, we need $\tan^3(2x) > 0$. This is equivalent to $\tan(2x) > 0$, so $2x \in (n\pi, (n + \frac{1}{2})\pi)$ for some integer n. Therefore, the domain of f is

$$\left\{ \left(\frac{n\pi}{2}, (\frac{n}{2} + \frac{1}{4})\pi \right) : n \text{ is any integer} \right\}.$$

On this domain, $\tan^3(2x)$ is positive and continuous. Since the composition of continuous functions is continuous, $f(x) = \log(\tan^3(2x))$ is continuous on the entire domain (described above).

(2) If $x \neq 1, -1$, then $|x| - 1$ is nonzero, so the quotient $\frac{x^2-1}{|x|-1}$ is continuous.

For $x = 1$, we check that

$$\begin{aligned}
\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2-1}{|x|-1} \\
&= \lim_{x\to 1} \frac{x^2-1}{x-1} \quad [\text{for } x \text{ close to } 1, x \text{ is positive so } |x| = x] \\
&= \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} \\
&= \lim_{x\to 1} (x+1) \\
&= 2.
\end{aligned}$$

Since this is equal to $f(1)$, f is continuous at 1.

Similarly,

$$\begin{aligned}\lim_{x\to-1} f(x) &= \lim_{x\to-1}\frac{x^2-1}{|x|-1}\\ &= \lim_{x\to-1}\frac{x^2-1}{-x-1} \quad [\text{for } x \text{ close to } -1,\ x \text{ is negative so } |x|=-x]\\ &= \lim_{x\to-1}\frac{(x+1)(x-1)}{-(x+1)}\\ &= \lim_{x\to-1} -(x-1)\\ &= 2.\end{aligned}$$

Since this is equal to $f(-1)$, f is continuous at -1.

Therefore, f is continuous on all of $\mathbb{R}$.

Exercise 3

(1)

$$\begin{aligned}\lim_{x\to0}\frac{\tan(\frac{1}{2}x^2)}{x} &= \lim_{x\to0}\frac{\sin(\frac{1}{2}x^2)}{x\cos(\frac{1}{2}x^2)}\\ &= \lim_{x\to0}\left(\frac{\frac{1}{2}x^2}{x\cos(\frac{1}{2}x^2)}\cdot\frac{\sin(\frac{1}{2}x^2)}{\frac{1}{2}x^2}\right)\\ &= \lim_{x\to0}\left(\frac{x}{2\cos(\frac{1}{2}x^2)}\cdot\frac{\sin(\frac{1}{2}x^2)}{\frac{1}{2}x^2}\right).\end{aligned}$$

As $x\to0$, $\frac{1}{2}x^2\to0$ as well, so $\lim_{x\to0}\frac{x}{2\cos(\frac{1}{2}x^2)} = \frac{0}{2\cos(0)} = 0$ and $\lim_{x\to0}\frac{\sin(\frac{1}{2}x^2)}{\frac{1}{2}x^2} = \lim_{t\to0}\frac{\sin t}{t} = 1$.

Therefore,

$$\lim_{x\to0}\frac{\tan(\frac{1}{2}x^2)}{x} = \left(\lim_{x\to0}\frac{x}{2\cos(\frac{1}{2}x^2)}\right)\cdot\left(\lim_{x\to0}\frac{\sin(\frac{1}{2}x^2)}{\frac{1}{2}x^2}\right) = 0\cdot1 = 0.$$

(2)

$$\begin{aligned}\lim_{x\to+\infty}\frac{(\sqrt{x}+x)(x-2)}{1+x\sqrt{x}} &= \lim_{x\to+\infty}\frac{(\sqrt{x}+x)(x-2)\cdot\frac{1}{x\sqrt{x}}}{(1+x\sqrt{x})\cdot\frac{1}{x\sqrt{x}}}\\ &= \lim_{x\to+\infty}\frac{\frac{\sqrt{x}+x}{\sqrt{x}}\cdot\frac{x-2}{x}}{\frac{1}{x\sqrt{x}}+1}\\ &= \lim_{x\to+\infty}\frac{(1+\sqrt{x})(1-\frac{2}{x})}{\frac{1}{x\sqrt{x}}+1}.\end{aligned}$$

As $x\to+\infty$, the term $1+\sqrt{x}\to+\infty$ as well, whereas $1-\frac{2}{x}\to1$ and $\frac{1}{x\sqrt{x}}+1\to1$. Therefore,

$$\lim_{x\to+\infty}\frac{(\sqrt{x}+x)(x-2)}{1+x\sqrt{x}} = +\infty.$$

(3) Since sin takes values between -1 and 1, we have

$$2^{-1} \le 2^{\sin x} \le 2^1 \implies \frac{1}{2}e^{-x^2} \le e^{-x^2}2^{\sin x} \le 2e^{-x^2}$$

for all x. Note that $\lim_{x\to+\infty} \frac{1}{2}e^{-x^2} = 0$ and $\lim_{x\to+\infty} 2e^{-x^2} = 0$, so the Squeeze Theorem implies that $\lim_{x\to+\infty} e^{-x^2}2^{\sin x} = 0$ and hence

$$\lim_{x\to+\infty} \cos\left(e^{-x^2}2^{\sin x}\right) = \cos(0) = 1.$$

(4)

$$\lim_{x\to 1^-} \left(\frac{x^3+2}{(x-1)(x-2)} - x\right) = \lim_{x\to 1^-} \left(\frac{1}{x-1}\cdot\frac{x^3+2}{x-2}\right) - 1$$

As $x \to 1^-$, $\frac{1}{x-1} \to -\infty$, whereas $\frac{x^3+2}{x-2} \to \frac{1+2}{1-2} = -3$. Therefore,

$$\lim_{x\to 1^-} \left(\frac{x^3+2}{(x-1)(x-2)} - x\right) = +\infty.$$

(5)

$$\begin{aligned}\lim_{x\to 5^+} \left(\frac{x-5}{x}\cdot\frac{e^x}{x^2-6x+5}\right) &= \lim_{x\to 5^+} \left(\frac{x-5}{x}\cdot\frac{e^x}{(x-1)(x-5)}\right)\\ &= \lim_{x\to 5^+} \left(\frac{1}{x}\cdot\frac{e^x}{x-1}\right)\\ &= \frac{1}{5}\cdot\frac{e^5}{5-1}\\ &= \frac{e^5}{20}.\end{aligned}$$

(6)

$$\begin{aligned}\lim_{x\to+\infty} \left(\sqrt{x^2+8x} - x\right) &= \lim_{x\to+\infty} \left(\sqrt{x^2+8x} - x\right)\cdot\frac{\sqrt{x^2+8x}+x}{\sqrt{x^2+8x}+x}\\ &= \lim_{x\to+\infty} \frac{(x^2+8x)-x^2}{\sqrt{x^2+8x}+x}\\ &= \lim_{x\to+\infty} \frac{8x}{\sqrt{x^2+8x}+x}\\ &= \lim_{x\to+\infty} \frac{8}{\frac{1}{x}(\sqrt{x^2+8x}+x)}\\ &= \lim_{x\to+\infty} \frac{8}{\sqrt{1+\frac{8}{x}}+1}\\ &= \frac{8}{\sqrt{1+0}+1}\\ &= 4.\end{aligned}$$

(7) As x approaches 0, $\sin\frac{\pi}{x}$ fluctuates between -1 and 1 infinitely often $\implies \sin^2\frac{\pi}{x}$ fluctuates between 0 and 1 $\implies \log_2(\sin^2\frac{\pi}{x}+1)$ fluctuates between $\log_2(0+1) = 0$ and $\log_2(1+1) = 1$. Therefore,

$$\lim_{x\to 0} \log_2\left(\sin^2\frac{\pi}{x}+1\right) \text{ does not exist.}$$

(8) Since cos takes values between -1 and 1, we have

$$-1 \le \cos\left(\sin\frac{1}{x}\right) \le 1 \implies -\sqrt{x} \le \sqrt{x}\cos\left(\sin\frac{1}{x}\right) \le \sqrt{x}$$

for all x. Note that $\lim_{x\to 0^+} -\sqrt{x} = 0$ and $\lim_{x\to 0^+}\sqrt{x} = 0$, so the Squeeze Theorem implies that

$$\lim_{x\to 0^+} \sqrt{x}\cos\left(\sin\frac{1}{x}\right) = 0.$$

(9)

$$\lim_{x\to -1} \sin\left(\frac{\pi(x-1)}{x^2+1}\right) = \sin\left(\frac{\pi(-1-1)}{(-1)^2+1}\right) = \sin\left(\frac{-2\pi}{2}\right) = \sin(-\pi) = 0.$$

(10)

$$\lim_{x\to\pi} \frac{\cos x + \sin x + 1}{x - 3\pi} = \frac{\cos\pi + \sin\pi + 1}{\pi - 3\pi} = \frac{-1+0+1}{-2\pi} = 0.$$

Exercise 4

Recall the definition:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

(1) For $f(x) = x^2$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h} \\ &= \lim_{h\to 0}\frac{2xh+h^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x. \end{aligned}$$

(2) For $f(x) = \dfrac{x}{x^2-4}$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x+h}{(x+h)^2-4} - \frac{x}{x^2-4}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\cdot\frac{(x+h)(x^2-4) - x[(x+h)^2-4]}{[(x+h)^2-4](x^2-4)} \\ &= \lim_{h\to 0}\frac{1}{h}\cdot\frac{(x^3+hx^2-4x-4h) - x[x^2+2xh+h^2-4]}{[(x+h)^2-4](x^2-4)} \\ &= \lim_{h\to 0}\frac{1}{h}\cdot\frac{-hx^2-4h-xh^2}{[(x+h)^2-4](x^2-4)} \\ &= \lim_{h\to 0}\frac{-x^2-4-xh}{[(x+h)^2-4](x^2-4)} \\ &= \frac{-x^2-4-0}{[(x+0)^2-4](x^2-4)} \\ &= -\frac{x^2+4}{(x^2-4)^2}. \end{aligned}$$

(3) For $f(x) = x + \sqrt{x}$,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h+\sqrt{x+h}) - (x+\sqrt{x})}{h} \\
&= \lim_{h\to 0} \frac{h+\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \left(1 + \frac{\sqrt{x+h}-\sqrt{x}}{h}\right) \\
&= 1 + \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= 1 + \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= 1 + \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= 1 + \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= 1 + \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= 1 + \frac{1}{\sqrt{x+0}+\sqrt{x}} \\
&= 1 + \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

Exercise 5

There are many possible answers, and one example is:

(a) $f(x) = 2 + \sin\frac{1}{x}, g(x) = \frac{1}{2+\sin\frac{1}{x}}$, so that $f(x)g(x)$ is identically 1.

(b) $f(x) = \frac{1}{x}, g(x) = -\frac{1}{x}$, so that $f(x) + g(x)$ is identically 0.