Calc-u-lose – HW4 – Part 1 – Solutions

1. $4x^2-6x+10=0 \Rightarrow x=\frac{1}{2\cdot 4}\left(-(-6)\pm\sqrt{(-6)^2-4\cdot 4\cdot 10}\right)$

$$= \frac{1}{8}\left(6\pm\sqrt{\underbrace{36-160}_{-124}}\right)$$

$\sqrt{-124}\notin\mathbb{R}$

$\Rightarrow$ x which solves the eq-n does not lie within $\mathbb{R}$ $\Leftrightarrow$ $\nexists$ real sol-ns.

2. $[a^4-4a^2+4]^{1/2}\equiv\sqrt{\underbrace{a^4-4a^2+4}_{(a^2-2)^2}}=|a^2-2|$

The result of the $\sqrt{\cdot}$ must be ≥ 0

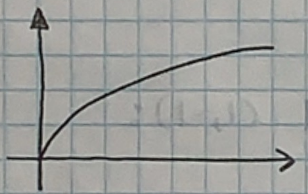
3. $\dfrac{\frac{x+1}{x^2-2x-3}}{\frac{x-3}{x+1}}=$

Note $x^2-2x-3=\Big(x-\underbrace{\tfrac{1}{2}\underbrace{(2+\underbrace{\sqrt{4+12}}_{4})}_{6}}_{3}\Big)\Big(x-\underbrace{\tfrac{1}{2}\underbrace{(2-\sqrt{16})}_{-2}}_{-1}\Big)$

$= (x-3)(x+1)$

$$=\frac{\frac{\cancel{x+1}}{(x-3)\cancel{(x+1)}}}{\frac{x-3}{x+1}}=\frac{x+1}{(x-3)^2}$$

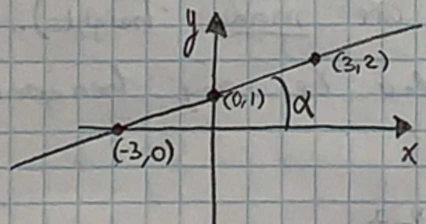
4. $(0,\infty)\ni x\mapsto\sqrt{x}$ is the f^n whose graph is sketched as (90° rotated half parabola)



5. $\mathbb{R}\ni x\mapsto \frac{x}{3}+1=:y(x)$ is a line. Let us find two points on it!

When $x=0$, $y(0)=1 \Rightarrow (0,1)\in\mathbb{R}^2$ one pt.

$x=3$, $y(3)=2 \Rightarrow (3,2)\in\mathbb{R}^2$ another pt.



Its slope is equal to $\frac{1}{3}$

$\Rightarrow$ angle α obeys $tg(\alpha)=\frac{1}{3}\Rightarrow \alpha=arctg(1/3)$.

We seek a line with slope $tg(\alpha+\frac{\pi}{2})$ (so that it's perpendicular) passing through $(3,2)$, so; a f^n g:

$g(x)=tg(\alpha+\frac{\pi}{2})x+b$ for some $b\in\mathbb{R}$.

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to find b, we plug in $(3,2)$:

$$2 \stackrel{!}{=} tg(\alpha+\tfrac{\pi}{2})3 + b \Rightarrow b = 2 - 3\,tg(\alpha+\tfrac{\pi}{2})$$

$$\Rightarrow g(x) = tg(\alpha+\tfrac{\pi}{2})x + 2 - 3\,tg(\alpha+\tfrac{\pi}{2})$$

$$= tg(\alpha+\tfrac{\pi}{2})(x-3) + 2$$

But what's $tg(\alpha+\frac{\pi}{2}) = tg(\arctg(1/3) + \pi/2) = ?$

Note $tg(\beta+\frac{\pi}{2}) \equiv \dfrac{\sin(\beta+\pi/2)}{\cos(\beta+\pi/2)} = \dfrac{\cos(\beta)}{-\sin(\beta)} \equiv -ctg(\beta)$

What is $ctg(\arctg(\gamma)) = ?$

$$ctg(\varepsilon) \equiv \frac{\cos(\varepsilon)}{\sin(\varepsilon)} = \frac{1}{tg(\varepsilon)}$$

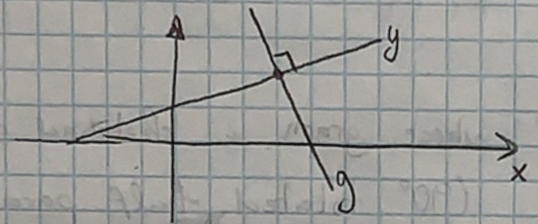
$\arctg \equiv tg^{-1}$

$$\Rightarrow ctg(\arctg(\gamma)) = \frac{1}{tg(\arctg(\gamma))} \equiv \frac{1}{\gamma}$$

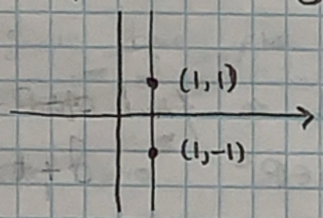
We find $tg(\alpha+\frac{\pi}{2}) = -ctg(\arctg(1/3))$

$$= -\frac{1}{1/3} = -3$$

$$\Rightarrow g(x) = -3(x-3) + 2 = -3x + 9 + 2 = -3x + 11$$



6. Line passing through $(1,1)$ and $(1,-1)$:



$\Rightarrow$ vertical line @ $x=1$

(not represented as a f^n $\mathbb{R} \ni x \mapsto ax+b$ since for each x there are <u>many</u> heights).

A vertical line is perpendicular to any horizontal line, e.g., $y=5$.

7. $$\underbrace{2(2x-3)}_{4x-6} + \underbrace{5(x+1)}_{5x+5} = 6x-7 \Leftrightarrow 9x - 1 = 6x - 7$$

$$\Leftrightarrow 3x = -6$$

$$\Leftrightarrow \boxed{x = -2}$$

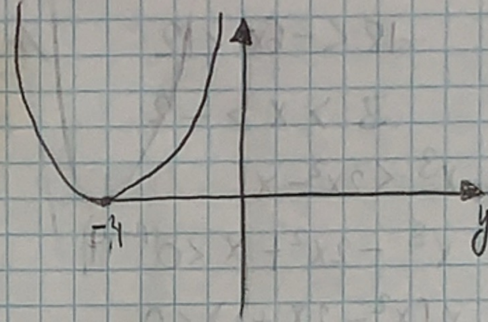
8. $x = 8$

$y^2x + yx^2 + 128 = 0$

$y^2\cancel{8} + y\underbrace{64}_{\cancel{8}\cdot 8} + \underbrace{128}_{\cancel{8}\cdot 8\cdot 2} = 0$

$y^2 + 8y + 16 = 0$

$y = \frac{1}{2}(-8 \pm \sqrt{\underbrace{64 - 4\cdot 16}_{=0}}) = -4$ //



9. $x^3 + 4x^2 + 3x = 0$

$x=0 \swarrow$ $\searrow x \neq 0$

$0 = 0$ ✓

$x^2 + 4x + 3 = 0$

$x = \frac{1}{2}(-4 \pm \sqrt{16-12}) = \frac{1}{2}(-4 \pm 2) = \nearrow -1, \searrow -3$

$\Rightarrow$ Largest sol-n is <u>zero</u>.

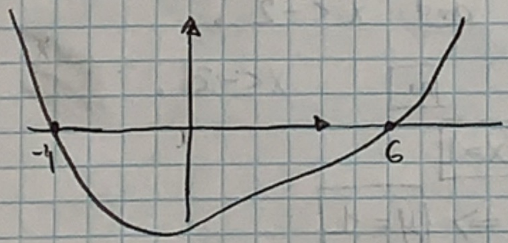
10. $\log(x) = 4 \Rightarrow \log(x^2) = 2\log(x) = 2\cdot 4 = 8.$

11. $\log_2(3xy^2) = \log_2(3) + \log_2(x) + 2\log_2(y).$

12. $\log(x^2 - 2x + 1) > \log(25)$) exp is strictly monotone increasing

$x^2 - 2x + 1 > 25$

$x^2 - 2x - 24 > 0$ Solves $x = \frac{1}{2}(\underbrace{-(-2)}_{2} \pm \underbrace{\sqrt{\underbrace{(-2)^2 + \underbrace{4\cdot 24}_{96}}_{100}}}_{10}) = \nearrow 6, \searrow -4$



$\Rightarrow$ $x < -4$ $\boxed{\text{or}}$ $x > 6$.

13. $h(x) := x^2,\ g(x) := x+1,\ f(x) := x+3$

$h(g(f(0))) = (g(f(0)))^2 = (f(0)+1)^2 = ((0+3)+1)^2 = 16.$

14. $h(x) = x^3,\ g(x) = x^2+1,\ f(x) = x+1$

$h(g(f(0))) + f(g(h(0))) = (1^2+1)^3 + ((0^2+1)+1) = 8+1+1 = 10.$

15. $f(x) := 3x^2 + 3,\quad f(f(a)) = 3(f(a))^2 + 3 = 3(3a^2+3)^2 + 3$

$= 3(9a^4 + 18a^2 + 9) + 3 = 27a^4 + 54a^2 + 30.$

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16. $-18 < -6x < 12 \quad / \cdot -6^{-1}$

$3 > x > -2$

17. $x^3 < 2x^2 - x$

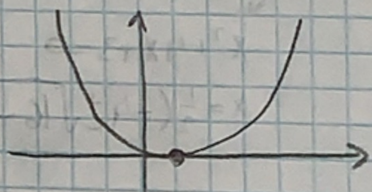
$x^3 - 2x^2 + x < 0$

$x(x^2 - 2x + 1) < 0$

$\Rightarrow$ either $x<0$ and $x^2-2x+1>0$ $\boxed{\text{or}}$

$x>0$ and $x^2-2x+1<0$

Solve $x^2-2x+1=0 \Leftrightarrow (x-1)^2=0 \Rightarrow x=1.$



$\Rightarrow x^2-2x+1>0$ if $x \neq 1.$

So we get $\boxed{x<0}$.

18. $(x+1)(x+2)>0 \Rightarrow x+1>0$ and $x+2>0$ $\boxed{\text{or}}$

$x+1<0$ and $x+2<0.$

$\Leftrightarrow x>-1$ and $x>-2$ $\boxed{\text{or}}$

$x<-1$ and $x<-2.$

$\Leftrightarrow x>-1$ $\boxed{\text{or}}$ $x<-2.$

19. $\begin{cases} 3x+4y=7 \\ 5x-4y=1 \end{cases} \Rightarrow 8x=8 \Rightarrow \boxed{x=1}$

$\Rightarrow 3+4y=7 \Rightarrow \boxed{y=1}.$

20. $\begin{cases} x-y=4 \\ 4x-y=1 \end{cases} \Rightarrow 3x=-3 \Rightarrow x=-1 \Rightarrow y=-5$

$\Rightarrow 3xy = 3(-1)(-5) = 15.$

21. $\begin{cases} y+3x+2=0 \\ y=x^2 \end{cases} \Rightarrow x^2+3x+2=0$

$x = \frac{1}{2}(-3 \pm \sqrt{9-4\cdot 2}) = \frac{1}{2}(-3 \pm 1) = \begin{cases} -1 \\ -2 \end{cases}$

22. $60° \longrightarrow \frac{60°}{360°} \cdot 2\pi = \frac{1}{6} \cdot 2\pi = \pi/3.$

23. $\sin(a+b)=1, \quad tg(a)=0$

$tg(b)=?$

$tg(a)=0 \Rightarrow \frac{\sin(a)}{\cos(a)}=0$

$\Rightarrow a=\pi n \ \exists\, n\in\mathbb{Z}$

$\sin(a+b)=1 \Leftrightarrow a+b=\frac{\pi}{2}+2\pi m$

$\exists\, m\in\mathbb{Z} \Rightarrow b=\frac{\pi}{2}+2\pi m-\pi n$

$tg(b)=$

$= tg(\frac{\pi}{2}+2\pi m-\pi n)$

$= tg(\frac{\pi}{2}-\pi n)$

$= \frac{\sin(\frac{\pi}{2}-\pi n)}{\cos(\frac{\pi}{2}-\pi n)}$

$= \frac{\sin(\pi/2)}{\cos(\pi/2)}$

$= tg(\pi/2)$

$= \infty.$

<u>Homework 4: Calculus 1 (SOLUTIONS)</u>

Continuity and its properties, the intermediate value theorem, introduction to derivatives

<u>Problem 1:</u>

<u>Part a:</u>

a) The goal is to find a value of c in [-2,4] such that:

$$f(c)=0=M$$

This is exactly the second conclusion of the Intermediate Value Theorem.
So, let's check that the "requirements" of the theorem are met.

<u>First,</u> the function is a polynomial and so is continuous everywhere and in particular is continuous on the interval $[-2,4]$.

<u>Second,</u> Now all that we need to do is verify that MM is between the function values as the endpoints of the interval. So,

$$f(-2)=1,\ f(4)=-167$$

Therefore, we have,

$$f(4)=-167<0<1=f(-2)$$

So, by the Intermediate Value Theorem there must be a number cc such that,

$$-2<c<4 \ \&\ f(c)=0$$

b) The problem is then asking us to show that there is a cc in $[-2,4]$ so that:

$$f(c)=0=M$$

First, the function is a sum of a polynomial (which is continuous everywhere) and a natural logarithm (which is continuous on $w>-25$ - *i.e.* where the argument is positive) and so is continuous on the interval [0,4].
Now all that we need to do is verify that MM is between the function values as the endpoints of the interval. So,

$$f(0)=-2.7726 \quad f(4)=3.6358$$

Therefore, we have,

$$f(0)=-2.7726<0<3.6358=f(4)$$

So, by the Intermediate Value Theorem there must be a number cc such that,

$$0<c<4 \ \&\ f(c)=0$$

c) The problem is then asking us to show that there is a c in $[-2,4]$so that,

$$f(c)=0=M$$

First, the function is a sum and difference of a polynomial and two exponentials (all of which are continuous everywhere) and so is continuous on the interval $[1,3]$.

Now all that we need to do is verify that MM is between the function values as the endpoints of the interval. So,

$$f(1)=23.7938 \quad f(3)=-190.5734$$

Therefore, we have,

$$f(3)=-190.5734<0<23.7938=f(1)$$

So, by the Intermediate Value Theorem there must be a number cc such that,

$$1<c<3 \ \& \ f(c)=0$$

Part b:

a) TRUE. It is an exact application of the intermediate value theorem (look at the previous problems)
b) FALSE: the intermediate value theorem indicates the existence of "at least" one real value of b that works. IT does not imply uniqueness
c) TRUE
d) FALSE
(c and d stem from questions a and b)

Problem 2:

This problem tests the thorough understanding of limits. Understanding what limits fundamentally are is necessary to understand continuity.

Please go back to the very definition of a limit to derive the solutions.

Propositions 2, 4, 5, 7 are true.

Problem 3:

There are multiple ways of proving the derivative of a product. (product rule). As long as the definition of a derivative is correctly used and the steps to get to the result are logical, give full credit. See the lecture notes Claim 8.14.

Problem 4:

a) Continuity has to be study for the value 0. (it is obviously continuous for all real non zero values)
   Sqrt(x) = $|x|$

If $x < 0$ then, $|x|$ = -x and thus $f(x) = x - 1$

Thus, $\lim_{x\to 0-} f(x) = -1$

If $x > 0$ then $f(x) = = x + 1$,

Thus, $\lim_{x\to 0+} f(x) = 1$

Therefore, the function is not continuous at zero (limits do not match)

b)

c) The limits when x goes to 0- and 0+ match.
   The limit of f when x goes to 0- is the limit of sin(x)/x so it is 1
   When x = 0, f(0)=1
   The limit of f when x goes to 0+ is the limit of x^2+1 so it is 1
   All of these values match, f is continuous at 0 (so continuous for all real values)

d) There is continuity if the limits at both 0.5- and 0.5+ match
   The limit of f as x goes to 0.5- is equal to f(0.5) which is 2/3
   The limit of f as x goes to 0.5+ is equal to 1+0.25 m

   Solve for m and get m = -4/3