Calculus 1 – Spring 2019 Section 2–HW5

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March 17, 2019

1 Review

Exercise 1. Simplify the following expressions:

1. $x, y \in \mathbb{R}$:

$$\begin{aligned} \frac{\frac{y}{x}-\frac{x}{y}}{\frac{1}{y}-\frac{1}{x}} &= \frac{\frac{y^2}{xy}-\frac{x^2}{xy}}{\frac{x}{xy}-\frac{y}{xy}} \\ &= \frac{y^2-x^2}{x-y} \\ &= \frac{(y-x)(y+x)}{x-y} \\ &= -x-y\,. \end{aligned}$$

2. $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} \frac{\left(\alpha^4\beta^{-3}\right)^2}{\alpha^3\beta^5} &= \frac{\alpha^8\beta^{-6}}{\alpha^3\beta^5} \\ &= \alpha^5\beta^{-11}\,, \end{aligned}$$

and

$$\begin{aligned} \frac{\frac{1}{\beta^2}+\alpha^2}{\frac{1}{\alpha^2}+\beta^2} &= \frac{\frac{1}{\beta^2}+\frac{\alpha^2\beta^2}{\beta^2}}{\frac{1}{\alpha^2}+\frac{\beta^2\alpha^2}{\alpha^2}} \\ &= \frac{\left(1+\alpha^2\beta^2\right)\alpha^2}{\beta^2\left(1+\alpha^2\beta^2\right)} \\ &= \frac{\alpha^2}{\beta^2}\,. \end{aligned}$$

Exercise 2. Solve for $x \in \mathbb{R}$:

$$\log_{10}(x-1) = 2\,.$$

We have the follow chain of equivalent equations, the first step results by applying $\exp_{10}$ on both sides of the equation and using the fact that $\exp_{10} \circ \log_{10} = \mathbb{1}$:

$$\begin{aligned} \log_{10}(x-1) &= 2 \\ x-1 &= 10^2 \\ x &= 101 \end{aligned}$$

Exercise 3. Factorize or complete the square for the following expressions for $\varepsilon, \delta \in \mathbb{R}$:

$$\begin{aligned} \varepsilon^4\delta-\delta^4\varepsilon &= \varepsilon\delta\left(\varepsilon^3-\delta^3\right) \\ &\quad \text{(Use the formula in the lecture notes in Example 6.34)} \\ &= \varepsilon\delta(\varepsilon-\delta)\left(\varepsilon^2+\varepsilon\delta+\delta^2\right) \\ &= \varepsilon\delta(\varepsilon-\delta)\left((\varepsilon+\delta)^2-\varepsilon\delta\right)\,. \end{aligned}$$

Note: it is not unequivocal which of these expressions is the "final" one that should be given as an answer, as the question stands now.

Exercise 4. For each of the following inequalities, provide an interval, open or closed, such that if x belongs to that interval, it satisfies the respective inequality:

1. $x(x-1)(x+2) > 0$.
   To satisfy this inequality, we need to have the product of three terms strictly positive. This happens if $x \in (-2,0) \cup (1,\infty)$. Indeed:
   (a) All are positive, i.e. $x > 0$ and $x > 1$ and $x > -2$ which all together implies that $x > 1$ (since $1 > 0 > -2$), which is the set $(1,\infty)$.
   (b) Two are negative and one is positive, which happens in three different cases:
      i. $x > 0$, $x < 1$ and $x < -2$, i.e. $(0,1) \cap (-\infty,-2) = \varnothing$, the empty set, i.e., this never happens.
      ii. $x < 0$, $x > 1$ and $x < -2$ i.e. $(-\infty,-2) \cap (1,\infty) = \varnothing$, the empty set, i.e., this also never happens.
      iii. $x < 0$, $x < 1$ and $x > -2$, i.e. $(-2,0)$.

2. $|x-3| < 4$.
   As we know, this inequality is equivalent to (Claim 6.1 in the lecture notes)

$$\begin{aligned} -4 <& \quad x-3 \quad && < 4 \\ 3-4 <& \quad x && < 3+4 \\ -1 <& \quad x && < 7 \end{aligned}$$

   i.e., this happens if $x \in (-1,7)$.

3. $x^2 < 3x + 8$.
   $\mathbb{R} \ni x \mapsto x^2 - 3x - 8$ is an upwards parabola, so we need to find out where it intersects the horizontal axis, possibly at two points, and our interval will be all points between these two. The solution to the equation

$$x^2 - 3x - 8 \quad = \quad 0$$

   is given by

$$\begin{aligned} x_{12} &= \frac{1}{2}\left(3 \pm \sqrt{9+4\cdot 8}\right) \\ &= \frac{3}{2} \pm \frac{\sqrt{41}}{2} \end{aligned}$$

   so that the interval is $x \in \left(\frac{3}{2} - \frac{\sqrt{41}}{2}, \frac{3}{2} + \frac{\sqrt{41}}{2}\right)$.

4. $\frac{2x-3}{x+1} \leq 1$.
   We assume that $x \neq -1$ (for otherwise the denominator becomes zero). Let us rewrite $2x - 3 = x + 1 + x - 4$ so that $\frac{2x-3}{x+1} = 1 + \frac{x-4}{x+1}$ and our inequality is equivalent to

$$\frac{x-4}{x+1} \quad \leq \quad 0$$

   which may happen if either numerator or denominator are negative, but not both. In other words, if ($x \geq 4$ and $x < -1$) or ($x \leq 4$ and $x > -1$). Since the first possibility is the empty set, we are left only with the second possibility, which is the interval $(-1,4]$.

Exercise 5. What equation is satisfied by all points on the plane $(x,y) \in \mathbb{R}^2$ which lie on the circle centered at the point $(a,b) \in \mathbb{R}^2$ and passing through the point $(c,d) \in \mathbb{R}^2$?

Solution. We don't know the radius of the circle. Let us give it a name none the less, $r \in (0,\infty)$. The equation for a circle of radius r centered at the point $(a,b) \in \mathbb{R}^2$ is given by the equation

$$(x-a)^2 + (y-b)^2 \quad = \quad r^2$$

(since it is the collection of all points whose distance from (a,b) is precisely r, and the equation above is merely the Pythagoras theorem. Now we find r via the constraint that the circle passes through the point $(c,d) \in \mathbb{R}^2$. Indeed, we plug in $(x,y) = (c,d)$ into the equation above to find that r must satisfy

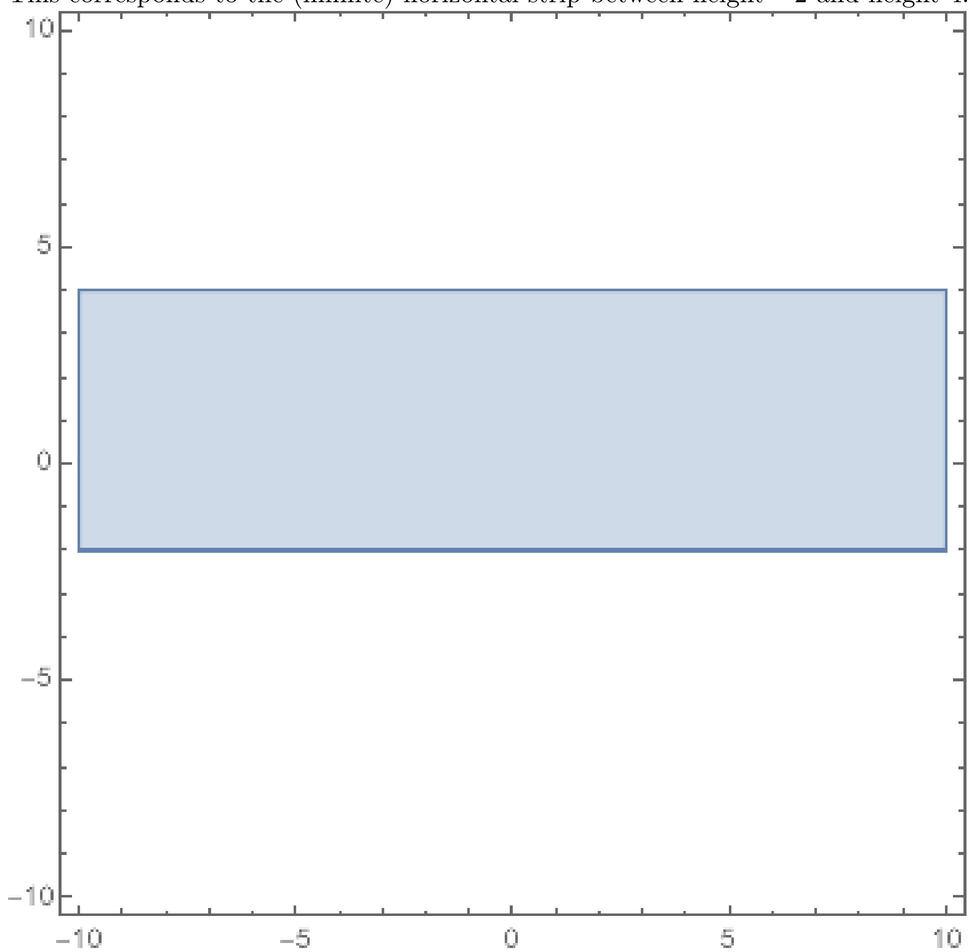
$$(c-a)^2 + (d-b)^2 \quad = \quad r^2 \,.$$

Hence the equation we seek is

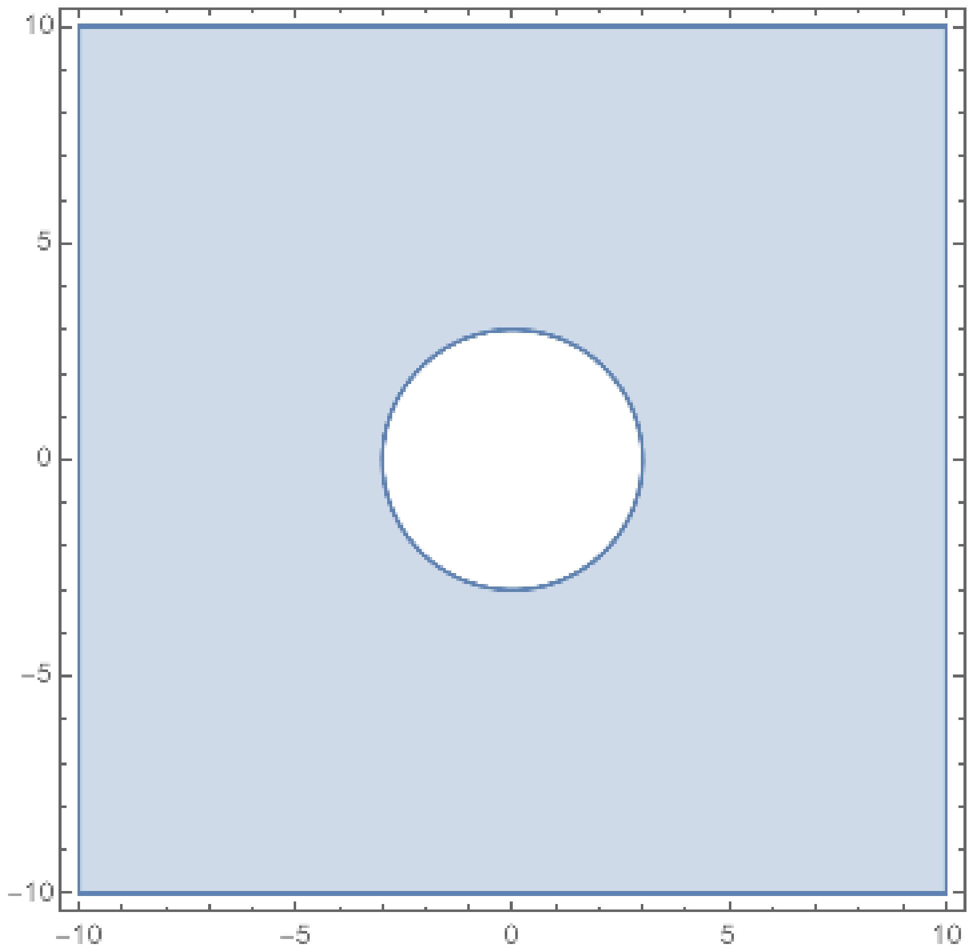
$$(x-a)^2 + (y-b)^2 \quad = \quad (c-a)^2 + (d-b)^2$$

Exercise 6. Sketch the region of $\mathbb{R}^2$ corresponding to all points $(x,y) \in \mathbb{R}^2$ which satisfy the following equalities or inequalities in x, y:

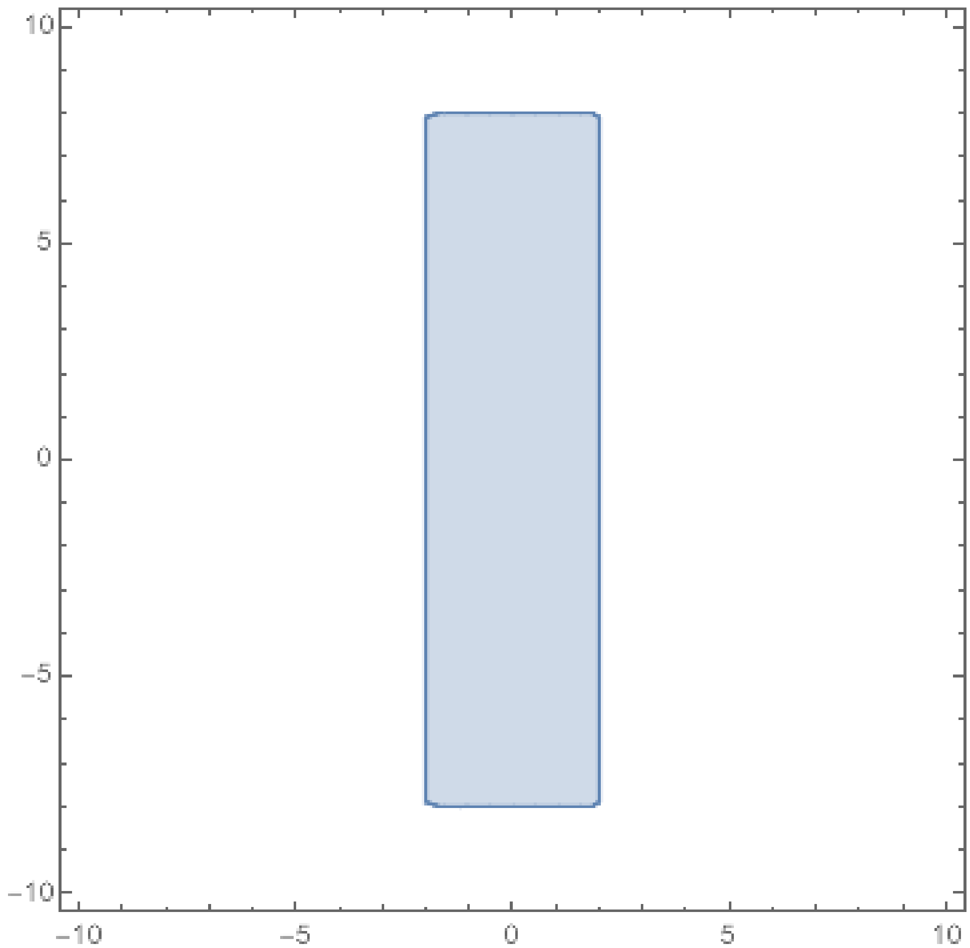
1. $-2 \le y < 4$.
   This corresponds to the (infinite) horizontal strip between height -2 and height 4.



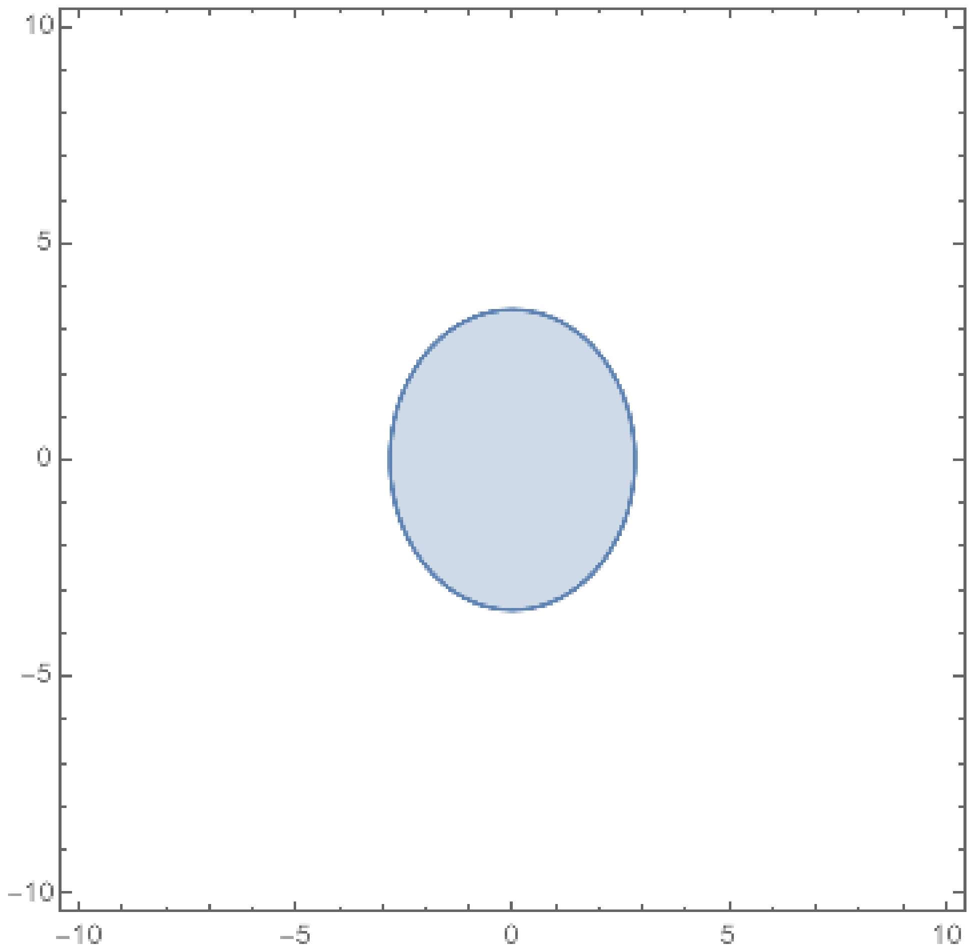
2. $x^2 + y^2 > 9$.
   This is the exterior of a circle of radius 3 about the origin:



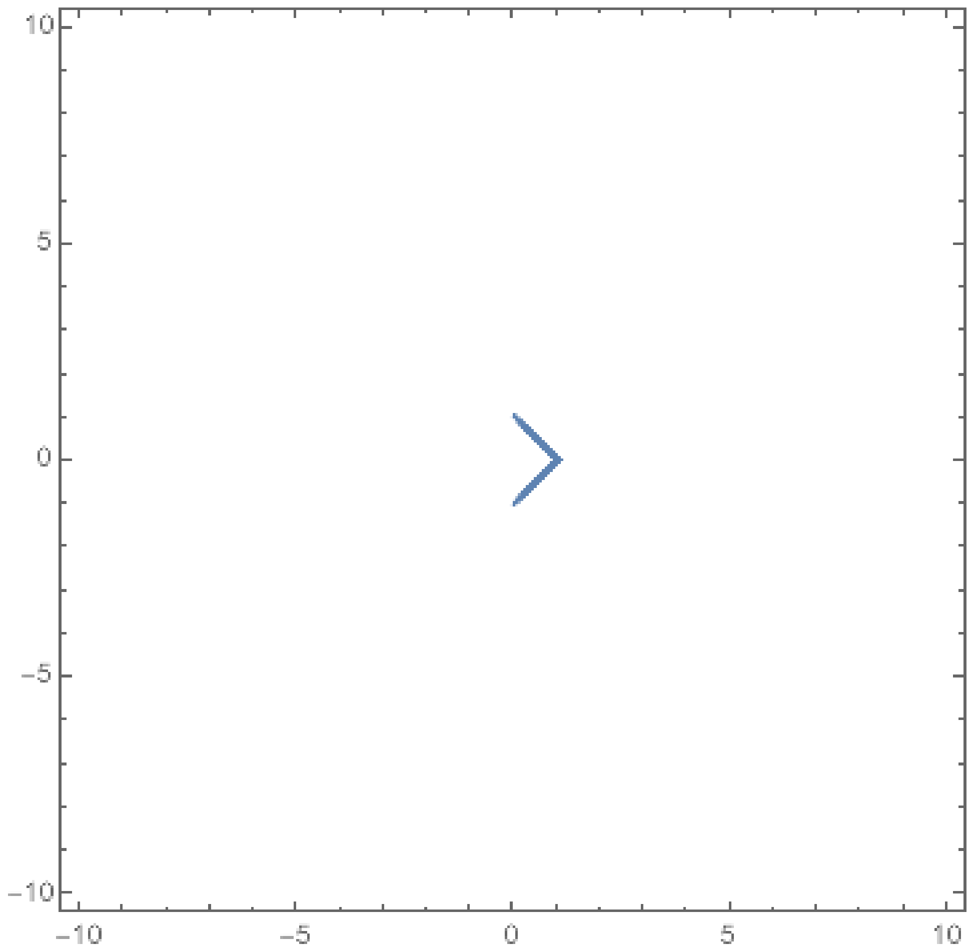
3. $|x| < 2$ and $|y| < 8$.
   The two inequalities are equivalent respectively to $-2 < x < 2$ and $-8 < y < 8$ which corresponds to the following rectangular area

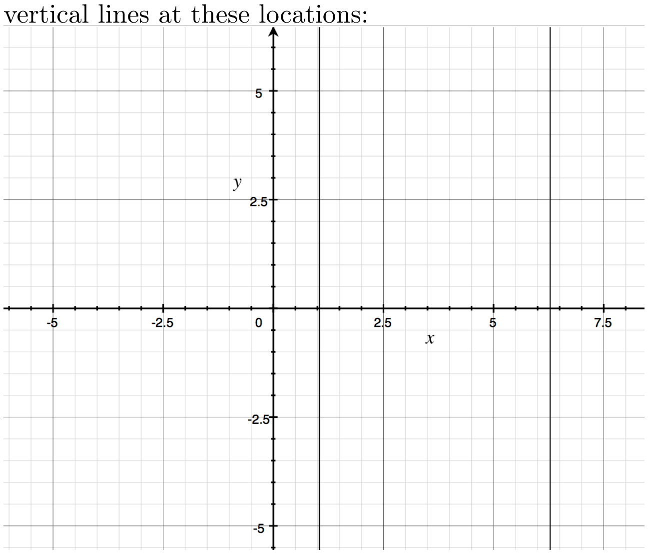


4. $\frac{x^2}{2} + \frac{y^2}{3} < 4$.
   This is the interior of an ellipse.



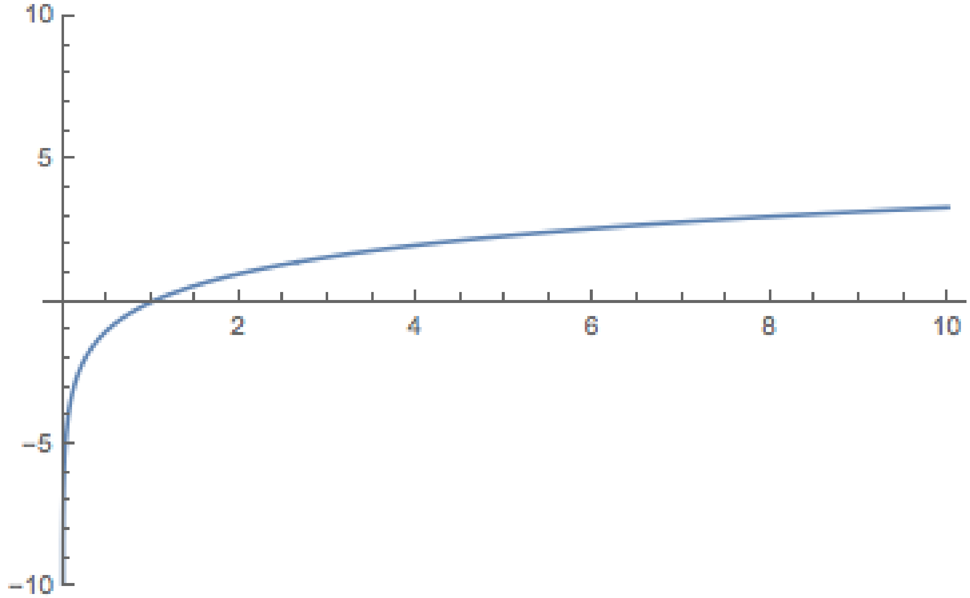
5. $|x| + |y| = 1$ and $x > 0$.
   The equation $|x| + |y| = 1$ actually corresponds to a rhombus about the origin. To see this, assume that we are considering only the first quadrant for example (so $x, y \geq 0$). Then the absolute values become redundant and we get $x + y = 1$ which is just the line $y = 1 - x$. One then sees the picture in the other three quadrants on a case by case basis. Now the restriction $x > 0$ merely means that we only look on the first and fourth quadrants, to finally obtain only half the rhombus:



6. $\sin(2x) = \sin(x)$ and $x \in [0, 2\pi]$.
   To solve this equation we need to rewrite $\sin(2x) = 2\sin(x)\cos(x)$. Within $[0, 2\pi]$, $\sin(x) = 0$ if $x = 0, \pi, 2\pi$. If $x \in (0, 2\pi) \setminus \{\pi\}$, then we get $2\cos(x) = 1$, or $\cos(x) = \frac{1}{2}$. This happens for $x \in [0, 2\pi]$ precisely at $x = \frac{\pi}{3}$.
   We conclude that the equation is satisfied either if $x = 0, \frac{\pi}{3}, 2\pi$. Since the constraints are independent of y, we get three vertical lines at these locations:
   

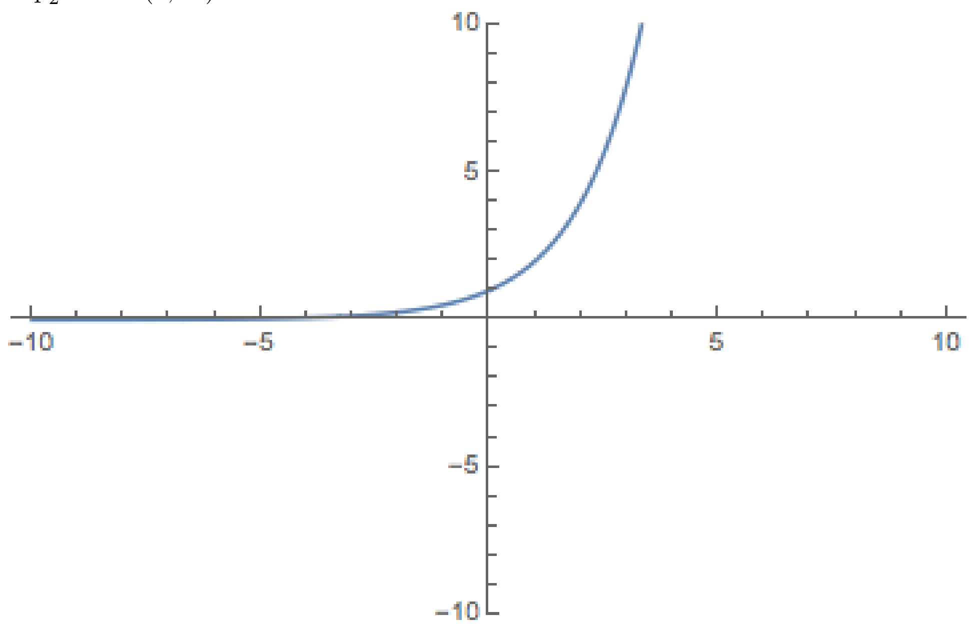
Exercise 7. Sketch the graph of the functions, clearly marking where they pass through the horizontal or vertical axes (if they do), and what happens at $\pm\infty$ (when relevant):

1. $\log_2 : (0, \infty) \to \mathbb{R}$.



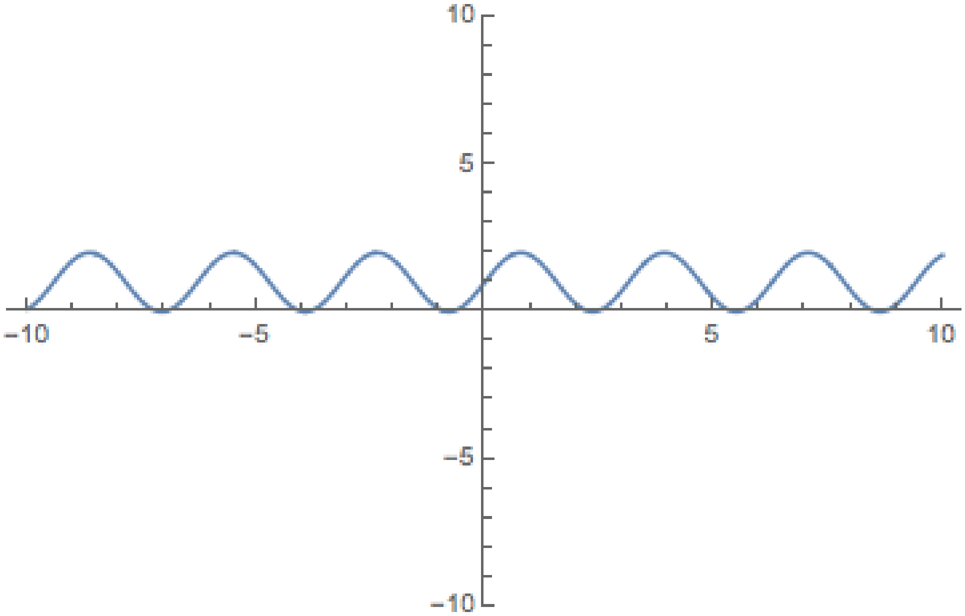
We have $\lim_{x\to 0} \log_2(x) = -\infty$, $\lim_{x\to\infty} \log_2(x) = +\infty$, $\log_2(1) = 0$. Note $\log_2(2) = 1$.

2. $\exp_2 : \mathbb{R} \to (0, \infty)$.



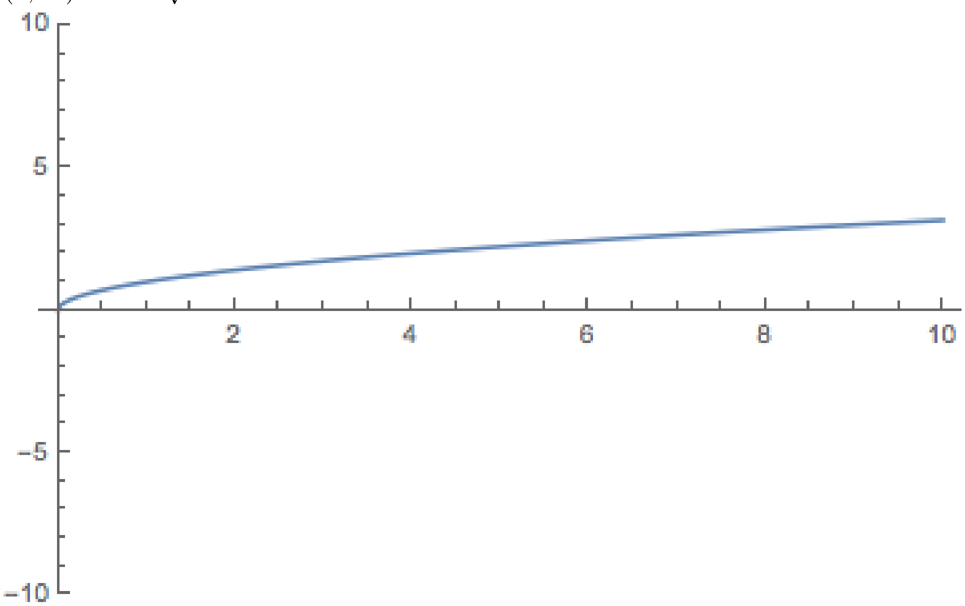
We have $\lim_{x\to -\infty} \exp_2(x) = 0$, $\lim_{x\to\infty} \exp_2(x) = +\infty$, $\exp_2(0) = 1$ and $\exp_2(x) > 0$ for *all* $x \in \mathbb{R}$.

3. $\mathbb{R} \ni x \mapsto 1 + \sin(2x) \in \mathbb{R}$.



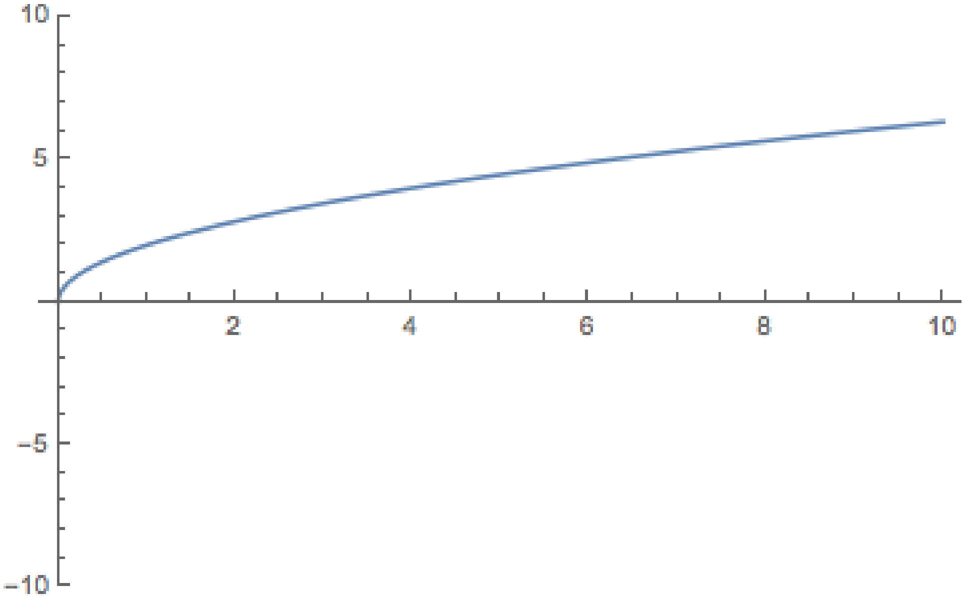
There are no limits at $\pm\infty$. The curve touches the horizontal axis whenever $\sin(2x) = -1$, i.e. whenever $x = \frac{1}{2}\left(-\frac{\pi}{2} + 2\pi n\right)$ for any $n \in \mathbb{Z}$. We note that at $x = 0$, we have height 1.

4. $(0, \infty) \ni x \mapsto \sqrt{x} \in \mathbb{R}$.



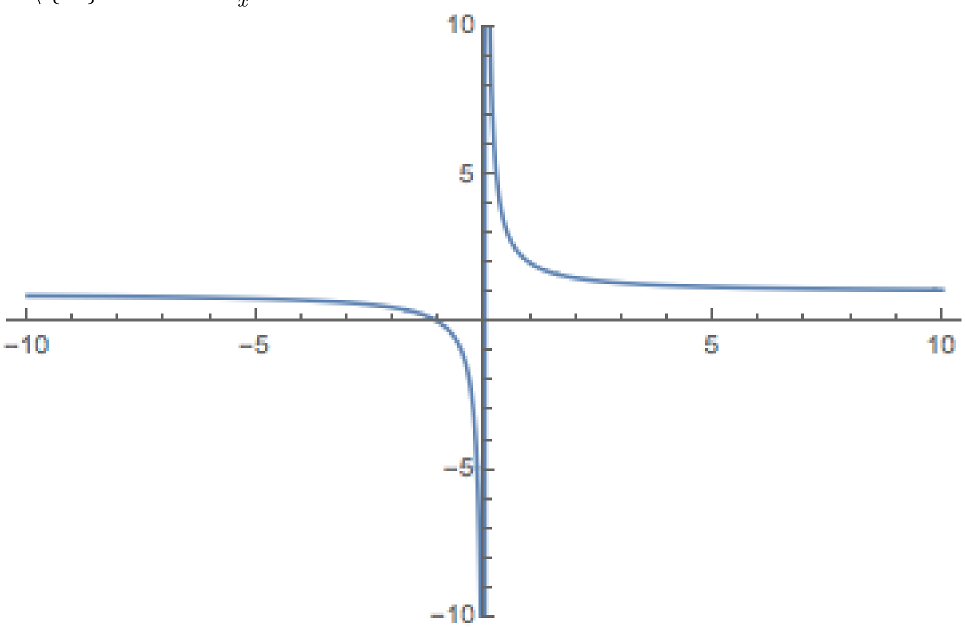
We have $\sqrt{0} = 0$, $\lim_{x\to\infty} \sqrt{x} = \infty$, and $\sqrt{x} > 0$ for all $x > 0$.

5. $(0, \infty) \ni x \mapsto 2\sqrt{x} \in \mathbb{R}$.



This is essentially the same graph, albeit somewhat stretched, but the same conclusions hold.

6. $\mathbb{R} \setminus \{ 0 \} \ni x \mapsto 1 + \frac{1}{x} \in \mathbb{R}$.



We have $\lim_{x \to -\infty} 1 + \frac{1}{x} = 1$, $\lim_{x \to \infty} 1 + \frac{1}{x} = 1$, $\lim_{x \to 0^-} 1 + \frac{1}{x} = -\infty$, $\lim_{x \to 0^+} 1 + \frac{1}{x} = +\infty$.
The curve crosses the horizontal axis at $x = -1$.
So this is a weird function (discontinuous at $x = 0$) where the height is decreasing from 1 on the left all the time, jumping at zero from $-\infty$ to $+\infty$, and then still going down all the time still always being *above* one.

Exercise 8. What are all values of $x \in [-10, 10]$ where the following expressions are zero? (restrict x when necessary)

1. $\log_{\mathrm{e}}(x)$? (Recall $\mathrm{e} \approx 2.718$ from the appendix of the lecture notes)
   The solution for $\log(x) = 0$ is $x = 1$.

2. $\sin(3x)$?
   To have $\sin(y) = 0$ we need $y \in \pi\mathbb{Z}$, so we have $3x = \pi n$ for any $n \in \mathbb{Z}$ in general, so $x = \frac{\pi}{3} n$ for $n \in \mathbb{Z}$ and we just must make sure that $\frac{\pi}{3} n \in [-10, 10]$, which means $n \in \left[-\frac{30}{\pi}, \frac{30}{\pi}\right] \approx [-9.54, 9.54]$. Since n has to be an integer, we get $n = -9, -8, \ldots, 8, 9$. Hence $x = -9\frac{\pi}{3}, -8\frac{\pi}{3}, \ldots, 8\frac{\pi}{3}, 9\frac{\pi}{3}$.

3. $\tan(x) + \sin(x)$?

The solution for

$$\begin{aligned}\tan(x)+\sin(x) &= 0\\ \sin(x)\left(\frac{1}{\cos(x)}+1\right) &= 0\\ \sin(x)\frac{1+\cos(x)}{\cos(x)} &= 0\end{aligned}$$

So either $\sin(x)=0$ or $\cos(x)=-1$ (and we must make sure that $\cos(x)\neq 0$).
$\sin(x)=0$ implies $x\in\pi\mathbb{Z}$ and $\cos(x)=-1$ implies $x\in\pi+2\pi\mathbb{Z}$ and $\cos(x)\neq 0$ implies $x\notin\frac{\pi}{2}+\pi\mathbb{Z}$. Hence the final answer is $x\in\pi\mathbb{Z}$ (since $\pi+2\pi\mathbb{Z}\subseteq\pi\mathbb{Z}$), in which case $\sin(x)=0$ always $\cos(x)$ is sometimes 1 and sometimes -1, but that doesn't matter.
When restricting to $x\in[-10,10]$, we find $x=-3\pi,-2\pi,-\pi,0,\pi,2\pi,3\pi$.

Exercise 9. Solve the following limits (possibly diverging to $\pm\infty$), or write "does not exist". Note you do not need to prove the existence from the definition.

1. $\lim_{n\to\infty}(\lim_{m\to\infty}\alpha)$ for some $\alpha\in\mathbb{R}$.
   The answer is α, since this is a constant in both m and n. The two limits may look intimidating but actually have no effect since nothing depends on the variables m or n.

2. $\lim_{n\to\infty}\sin\left(\frac{\pi}{2}n\right)$. Note this is the limit of a *sequence*, i.e. $n\in\mathbb{N}$ here.
   When $n=1,2,3,4,5,\dots$ we have $\frac{\pi}{2}n=\frac{\pi}{2},\pi,3\frac{\pi}{2},2\pi,5\frac{\pi}{2},\dots$ so that $\sin\left(n\frac{\pi}{2}\right)=1,0,-1,0,1,\dots$ and due to these oscillations, the limit does not exist.

3. $\lim_{x\to-\infty}\exp_a(x)$ for $a>1$.
   As we saw earlier, $\lim_{x\to-\infty}\exp_a(x)=0$. Recall $\exp_a(x)\equiv a^x$. When $x<0$, this equals $\frac{1}{a^{|x|}}$, i.e., as $x\to-\infty$, we are taking larger and larger powers of a number, a, which is larger than 1, so $a^{|x|}$ becomes larger and larger, and hence $\frac{1}{a^{|x|}}$ becomes smaller and smaller.

4. $\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right)$.
   When we have double limits, we need to proceed according evaluate the inner limit first, as if the variable of the outer limit is a constant, and only afterwards evaluate the outer limit. Hence,

   $$\lim_{n\to\infty}\frac{m}{m+n} = 0$$

   since

   $$\begin{aligned}\lim_{n\to\infty}\frac{m}{m+n} &= m\lim_{n\to\infty}\frac{1}{m+n}\\ &= m\cdot 0\\ &= 0\end{aligned}$$

   and then when we take the limit $m\to\infty$, there is already no m dependence and we are just left with zero.

5. $\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right)$.
   We do the same thing, but now the order of the limits is reversed, so that

   $$\begin{aligned}\lim_{m\to\infty}\frac{m}{m+n} &= \lim_{m\to\infty}\left(\frac{m+n}{m+n}-\frac{n}{m+n}\right)\\ &= \lim_{m\to\infty}\left(1-\frac{n}{m+n}\right)\\ &= 1-\lim_{m\to\infty}\frac{n}{m+n}\\ &= 1-0\\ &= 1\end{aligned}$$

   and now again there is no n dependence, so that the result is just 1.
   This and the previous double limit together show that the order in which we take limits matters!

6. $\lim_{n\to\infty}\sum_{k=0}^{n} ar^k$ for $a\in\mathbb{R}$ and $r<1$. You may consult the appendix in the lecture notes if you find the symbol $\sum$ unfamiliar; use the formula $\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r}$.
We have

$$\begin{aligned}
\sum_{k=0}^{n} ar^k &= a\sum_{k=0}^{n} r^k \\
&\quad \text{(Rename variable } m := k+1) \\
&= a\sum_{m=1}^{n+1} r^{m-1} \\
&= a\left(\sum_{k=1}^{n} r^{k-1} + r^n\right) \\
&\quad \text{(Use formula from hint)} \\
&= a\left(\frac{1-r^n}{1-r} + r^n\right)
\end{aligned}$$

Now we are in a position to take the $n\to\infty$ limit. Crucially, we use the fact that $r<1$, so that $\lim_{n\to\infty} r^n = 0$ (see Claim 6.16 in the lecture notes) and we find

$$\lim_{n\to\infty}\sum_{k=0}^{n} ar^k = \frac{a}{1-r}.$$

7. $\lim_{n\to\infty}\sum_{k=0}^{n} ar^k$ for $a\in\mathbb{R}$ and $r>1$.
Following the same procedure, we find

$$\begin{aligned}
\sum_{k=0}^{n} ar^k &= a\left(\frac{1-r^n}{1-r} + r^n\right) \\
&= a\left(\frac{1-r^n+r^n-r^{n+1}}{1-r}\right) \\
&= a\left(\frac{1-r^{n+1}}{1-r}\right)
\end{aligned}$$

However, now, since $r>1$, $\lim_{n\to\infty} r^{n+1} = \infty$, so that the limit diverges to $-\infty$.

2 Exercises pertaining to new material

Exercise 10. Calculate the derivative of the following functions (no proof necessary) and restrict the domain of the derivative if necessary:
In the various exercises one has to restrict the domain so that one never divides by zero and so on. We don't include this discussion here.

1. $x \mapsto \frac{\sec(x)}{1+\tan(x)}$.
We have using the quotient rule

$$\left(\frac{\sec}{1+\tan}\right)' = \frac{\sec'(1+\tan) - \sec(1+\tan)'}{(1+\tan)^2}$$

Now we need

$$\begin{aligned}
\sec' &\equiv \left(\frac{1}{\cos}\right)' \\
&\quad \text{(Use chain rule)} \\
&= -\frac{1}{\cos^2}\cos' \\
&= \frac{1}{\cos^2}\sin \\
&= \frac{\tan}{\cos}
\end{aligned}$$

and we already know from the lecture that $\tan' = \frac{1}{\cos^2} \equiv \sec^2$. Hence

$$\begin{aligned}\left(\frac{\sec}{1+\tan}\right)' &= \frac{\sec'(1+\tan)-\sec(1+\tan)'}{(1+\tan)^2}\\ &= \frac{\frac{\tan}{\cos}(1+\tan)-\sec\sec^2}{(1+\tan)^2}\\ &= \frac{\tan\sec(1+\tan)-\sec^3}{(1+\tan)^2}\\ &= \frac{\sec\left(\tan(1+\tan)-\sec^2\right)}{(1+\tan)^2}\end{aligned}$$

and

$$\begin{aligned}\tan(1+\tan)-\sec^2 &= \sin\sec(1+\sin\sec)-\sec^2\\ &= \sin\sec+\sin^2\sec^2-\sec^2\\ &= \sec^2\left(\sin^2-1\right)+\sin\sec\\ &= \sec^2\left(-\cos^2\right)+\sin\sec\\ &= -1+\sin\sec\\ &= \tan-1\end{aligned}$$

So the final answer is

$$\left(\frac{\sec}{1+\tan}\right)' = \frac{\sec(\tan-1)}{(1+\tan)^2}\,.$$

2. $x \mapsto (\cos(x))^2$, $x \mapsto \cos\left(x^2\right)$ and $x \mapsto \cos(\cos(x))$.
   We proceed in steps
   (a) We have using the power rule and the chain rule

   $$\begin{aligned}\left(x \mapsto \cos(x)^2\right)' &= 2\cos(x)\cos'(x)\\ &= -2\cos(x)\sin(x)\\ &= -\sin(2x)\,.\end{aligned}$$

   (b) Now we use the chain rule first and then the power rule

   $$\begin{aligned}\left(x \mapsto \cos\left(x^2\right)\right)' &= -\sin\left(x^2\right)\left(x \mapsto x^2\right)'\\ &= -\sin\left(x^2\right)2x\\ &= -2x\sin\left(x^2\right)\,.\end{aligned}$$

   (c) Finally, one could interpret two applications of cos (via composition) as $\cos^2$ too, which is why this is the third option given. Hence using the chain rule alone we get

   $$\begin{aligned}(\cos\circ\cos)' &= (\cos'\circ\cos)\cos'\\ &= (-\sin\circ\cos)(-\sin)\\ &= (\sin\circ\cos)\sin\end{aligned}$$

   which can also be written as

   $$(x \mapsto \cos(\cos(x)))' = \sin(\cos(x))\times\sin(x)\,.$$

3. tan.
   We saw in the lecture (Example 8.30) that this is equal to sec.

4. cot.
   We have
$$\begin{aligned}\cot &\equiv \frac{\cos}{\sin}\\ &= \frac{1}{\tan}\end{aligned}$$
   and so we may use the previous result plus the chain rule to get
$$\begin{aligned}\cot' &= -\frac{1}{\tan^2}\tan'\\ &= -\frac{1}{\tan^2}\sec\\ &= -\frac{\cos^2}{\sin^2}\frac{1}{\cos}\\ &= -\frac{1}{\sin^2}\,.\end{aligned}$$

5. $\mathbb{R} \ni x \mapsto (e^x)^2$.
   We re-write this as $(e^x)^2 \equiv \exp(x)^2 = \exp(x)\exp(x) = \exp^2$. Hence the chain rule and the power rule gives
$$\begin{aligned}\left(\exp^2\right)' &\equiv 2\exp\exp'\\ &= 2\exp\exp\\ &= 2\exp^2\,.\end{aligned}$$
   Another way to view this result is using the rules of exponents, which tell us that $(e^x)^2 = e^{2x}$ so that $\left(e^{2x}\right)' = e^{2x}2$.

6. $\mathbb{R} \ni x \mapsto e^{e^{e^{e^x}}}$.
   This exercise is just really annoying, what is going on here? Just many many applications of the chain rule. Let us count:
$$\begin{aligned}e^{e^{e^{e^x}}} &= \exp(\exp(\exp(\exp(x))))\\ &= (\exp\circ\exp\circ\exp\circ\exp)(x)\end{aligned}$$
   and so using the chain rule four times we get
$$\begin{aligned}(\exp\circ\exp\circ\exp\circ\exp)' &= (\exp'\circ\exp\circ\exp\circ\exp)(\exp\circ\exp\circ\exp)'\\ &= (\exp\circ\exp\circ\exp\circ\exp)(\exp'\circ\exp\circ\exp)(\exp\circ\exp)'\\ &= (\exp\circ\exp\circ\exp\circ\exp)(\exp\circ\exp\circ\exp)(\exp'\circ\exp)\exp\\ &= (\exp\circ\exp\circ\exp\circ\exp)(\exp\circ\exp\circ\exp)(\exp\circ\exp)\exp\end{aligned}$$
   Rewritten with the argument, this becomes
$$\begin{aligned}\left(e^{e^{e^{e^x}}}\right)' &= e^{e^{e^{e^x}}}e^{e^{e^x}}e^{e^x}e^x\\ &\quad\text{(Factor together)}\\ &= e^{x+e^x+e^{e^x}+e^{e^{e^x}}}\,.\end{aligned}$$

7. $\mathbb{R} \ni x \mapsto 5$.
   The derivative of any constant function is zero.

8. $(0,\infty) \ni x \mapsto \sqrt[3]{x}$.
   This is solved in the lecture notes in Example 8.25.

9. $\mathbb{R}\setminus\{0\}\ni x\mapsto\frac{5x^2+10x+20}{80x^{100}}$.
We apply the quotient rule to get

$$\begin{aligned}\left(x\mapsto\frac{5x^2+10x+20}{80x^{100}}\right)' &= \frac{\left(x\mapsto 5x^2+10x+20\right)'\left(x\mapsto 80x^{100}\right)-\left(x\mapsto 5x^2+10x+20\right)\left(x\mapsto 80x^{100}\right)'}{\left(x\mapsto 80x^{100}\right)^2}\\ &= x\mapsto\frac{(10x+10)\,80x^{100}-\left(5x^2+10x+20\right)8000x^{99}}{6400x^{200}}\\ &= x\mapsto\left(\frac{10x+10}{80x^{100}}-25\frac{x^2+2x+4}{4x^{101}}\right).\end{aligned}$$

10. $\mathbb{R}\setminus\{0\}\ni x\mapsto\frac{1}{|x|}$ whenever possible.
With $r(x):=\frac{1}{x}$ and $a(x):=|x|$ we have $x\mapsto\frac{1}{|x|}=r\circ a$, and so

$$(r\circ a)' = (r'\circ a)\,a'$$

and recall $r'=-r^2$ and $a'=s$ where

$$s(x) \equiv \begin{cases}1 & x>0\\ -1 & x<0\end{cases}.$$

Hence

$$(r\circ a)' = -\frac{s}{a^2}$$

with the argument this becomes

$$\begin{aligned}\left(x\mapsto\frac{1}{|x|}\right)' &= -\frac{s(x)}{|x|^2}\\ &\quad\left(\text{Rewrite } s(x)=\frac{|x|}{x}\text{ for all } x\neq 0\right)\\ &= -\frac{|x|}{x\,|x|^2}\\ &= -\frac{1}{x\,|x|}.\end{aligned}$$

11. $\exp\circ\cos$.
This is an easy application of the chain rule

$$\begin{aligned}(\exp\circ\cos)' &= (\exp'\circ\cos)\cos'\\ &= (\exp\circ\cos)(-\sin)\\ &= -(\exp\circ\cos)\sin\,.\end{aligned}$$

Exercise 11. If f is differentiable then f' is a new function, on which we may *yet again* ask whether f' itself differentiable. If it is, then we can calculate in turn its own derivative, $(f')'$ which is called *the second derivative* (denoted by f'') of f. Calculate the second derivative of the following functions:

1. $\mathbb{R}\ni x\mapsto x$.
Since $\mathbb{1}'=1$, $\mathbb{1}''=1'=0$.

2. $\mathbb{R}\ni x\mapsto x^2$.
We have $\left(x\mapsto x^2\right)'=x\mapsto 2x$ whose derivative is just $x\mapsto 2$.

3. $\mathbb{R}\setminus\{0\}\ni x\mapsto\frac{1}{x}$.
We know that, with $r(x)\equiv\frac{1}{x}$, $r'=-r^2$, so that $r''=\left(-r^2\right)=-2rr'=-2r\left(-r^2\right)=2r^3$. I.e.

$$\left(x\mapsto\frac{1}{x}\right)'' = x\mapsto\frac{2}{x^3}.$$

Exercise 12. Determine whether the following functions are increasing or decreasing by examining the sign of their derivative. Recall that if the derivative was positive at some point, then the function was increasing and if it was negative the function was decreasing.

1. exp.
   Since $\exp' = \exp$ and $\exp > 0$, the function exp is always increasing.

2. log.
   We have $\log' = r$ with $r(x) \equiv \frac{1}{x}$ for all $x > 0$. Since $\mathrm{im}(r) \subseteq (0, \infty)$, i.e., $r(x) > 0$ for all $x > 0$, we have log increasing on $(0, \infty)$.

3. $x \mapsto x^3$.
   $\left(x \mapsto x^3\right) = 3x^2$ which is *always* positive since x^2 is always positive. Hence $x \mapsto x^3$ is always increasing.

4. $(0, \infty) \ni x \mapsto \sqrt{x}$.
   We have $(x \mapsto \sqrt{x})' = x \mapsto \frac{1}{2\sqrt{x}}$ which is itself always positive, so the square root is always increasing.

Exercise 13. Use the so-called *the hospital rule,* when appropriate (sometimes you can just proceed directly) in order to evaluate the following limits.

1. For any $n \in \mathbb{N}$, $\lim_{x\to\infty} \frac{x^n}{e^x}$.
   We have, by the algebra of the limits, $\lim_{x\to\infty} x^n = \left(\lim_{x\to\infty} x\right)^n = \infty$. For the denominator, we have $\lim_{x\to\infty} e^x = \infty$, so the limit is of the form $\frac{\infty}{\infty}$, and perhaps we can apply l'Hospital's rule, if the limit of the quotient of the derivatives exists. This corresponds to

$$\lim_{x\to\infty} \frac{(x^n)'}{(e^x)'} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}$$

   This is still of the form $\frac{\infty}{\infty}$, unless $n = 1$, in which case $x^{n-1} = 1$, and we get $\lim_{x\to\infty} \frac{n}{e^x} = n \lim_{x\to\infty} \frac{1}{e^x} = n \cdot 0 = 0$. This suggests we can apply l'Hospital's rule iteratively n-times until we get x-dependence in the numerator of the form $x^0 = 1$. In the denominator we'll always get the same thing, since $\exp' = \exp$. Hence the result is zero, after n-applications of the l'Hospital rule.
   If one wants to avoid l'Hospital rule, then one could also have concluded this directly, by appealing to the intuitive fact that $x \mapsto x^n$ grows *much slower* than $x \mapsto e^x$ as $x \to \infty$. To see this, we need to study bounds on the logarithm:

$$\begin{aligned} \frac{x^n}{\exp(x)} &= \frac{1}{x^{-n}\exp(x)} \\ &= \frac{1}{\exp(x + \log(x^{-n}))} \\ &= \frac{1}{\exp(x - n\log(x))} \end{aligned}$$

   Now when we take $\lim_{x\to\infty}$, we use the continuity of exp to push the limit through, and get

$$\lim_{x\to\infty} \frac{x^n}{\exp(x)} = \frac{1}{\exp\left(\lim_{x\to\infty}(x - n\log(x))\right)}$$

   and now we can use the bound on the logarithm we established before (from the practice midterm 1)

$$\log(y) \leq y - 1$$

   If we plug in $y = \frac{x}{2n}$ then we find

$$\begin{aligned} \log(x) &= \log\left(2n\frac{x}{2n}\right) \\ &= \log(2n) + \log\left(\frac{x}{2n}\right) \\ &\leq \log(2n) + \frac{x}{2n} - 1 \\ &\leq \log(2n) + \frac{x}{2n} \end{aligned}$$

and hence

$$\begin{aligned} x - n\log(x) &\geq x - n\left(\log(2n) + \frac{x}{2n}\right) \\ &= x - \frac{1}{2}x - n\log(2n) \\ &= \frac{1}{2}x - n\log(2n) \end{aligned}$$

Hence when we take $\lim_{x\to\infty}$ of both sides of this inequality (lecture notes Remark 6.15) we find

$$\begin{aligned} \lim_{x\to\infty}(x - n\log(x)) &\geq \lim_{x\to\infty}\left(\frac{1}{2}x - n\log(2n)\right) \\ &= \left(\lim_{x\to\infty}\frac{1}{2}x\right) - n\log(2n) \\ &= \infty - n\log(2n) \\ &= \infty\,. \end{aligned}$$

Now $\exp(\infty) \to \infty$ and $\frac{1}{\infty} \to 0$, so we find the same result as with l'Hospital's rule, but we had to work much harder with the estimates on log!

2. $\lim_{x\to 3}\frac{\cos(x)\log(x-3)}{\log(\mathrm{e}^x-\mathrm{e}^3)}$.
   The cos is just a distraction, we can ignore it first. Then the limit is of the form $\frac{\log(0)}{\log(0)} \sim \frac{-\infty}{-\infty}$, so maybe we can use the Hospital rule. Let us consider the quotient of the derivatives:

$$\begin{aligned} \lim_{x\to 3}\frac{\log(x-3)}{\log(\mathrm{e}^x-\mathrm{e}^3)} &\stackrel{?}{=} \lim_{x\to 3}\frac{\frac{1}{x-3}}{\frac{1}{\mathrm{e}^x-\mathrm{e}^3}\mathrm{e}^x} \\ &= \lim_{x\to 3}\mathrm{e}^{-x}\frac{\mathrm{e}^x-\mathrm{e}^3}{x-3} \\ &= \mathrm{e}^{-3}\lim_{x\to 3}\frac{\mathrm{e}^x-\mathrm{e}^3}{x-3} \end{aligned}$$

   this last limit is again of the form $\frac{0}{0}$, so we try l'Hospital on it again

$$\begin{aligned} \lim_{x\to 3}\frac{\mathrm{e}^x-\mathrm{e}^3}{x-3} &\stackrel{?}{=} \lim_{x\to 3}\frac{\mathrm{e}^x}{1} \\ &= \mathrm{e}^3 \end{aligned}$$

   this last limit exists, so we find all together that the two applications of l'Hospital's rule were justified and

$$\begin{aligned} \lim_{x\to 3}\frac{\cos(x)\log(x-3)}{\log(\mathrm{e}^x-\mathrm{e}^3)} &= \left(\underbrace{\lim_{x\to 3}\cos(x)}_{=\cos(3)}\right)\times\left(\underbrace{\lim_{x\to 3}\frac{\log(x-3)}{\log(\mathrm{e}^x-\mathrm{e}^3)}}_{=1}\right) \\ &= \cos(3)\,. \end{aligned}$$

3. $\lim_{x\to 0}\frac{\mathrm{e}^x-1}{\sin(x)}$.
   This limit is of the form $\frac{0}{0}$, so maybe we can apply l'Hospital's rule on it.

$$\begin{aligned} \lim_{x\to 0}\frac{\mathrm{e}^x-1}{\sin(x)} &\stackrel{?}{=} \lim_{x\to 0}\frac{\mathrm{e}^x}{\cos(x)} \\ &= \frac{1}{1} \\ &= 1\,. \end{aligned}$$

4. $\lim_{x\to 0}\frac{\tan(px)}{\tan(qx)}$ for two constants $p, q \in \mathbb{R}$.
   We have $\tan(0) = 0$ so that the limit is of the form $\frac{0}{0}$ and maybe we can apply l'Hospital rule. Note that

$$(x \mapsto \tan(px))' = x \mapsto \frac{p}{\cos(px)}$$

so that

$$\lim_{x\to 0}\frac{\tan(px)}{\tan(qx)} \overset{?}{=} \lim_{x\to 0}\frac{\frac{p}{\cos(px)}}{\frac{q}{\cos(qx)}}$$

but now $\cos(0)=1$, so we have the limit indeed exists and equals $\frac{p}{q}$:

$$\lim_{x\to 0}\frac{\tan(px)}{\tan(qx)} = \frac{p}{q}\,.$$

5. $\lim_{x\to 0}\frac{\log(x)}{x}$ (restrict x as necessary to make the logarithm make sense).
   We have $\log(0)\to-\infty$ and $x\to 0$, so the limit is of the form $\frac{-\infty}{0}$ which is *not* appropriate for l'Hospital's rule. We have the bounds (see the practice midterm or the midterm 1; Definition 10.2 in the lecture notes)

$$1-\frac{1}{x}\leq \log(x)\leq x-1$$

   Hence we have

$$\frac{1}{x}-\frac{1}{x^2} \leq \frac{\log(x)}{x} \leq 1-\frac{1}{x}$$

   where we note $\frac{1}{x}-\frac{1}{x^2}=\frac{1}{x}\left(1-\frac{1}{x}\right)$. When we now send $x\to 0$ (but only positive values of x, so that the logarithm makes sense) then

$$\frac{1}{x}\left(1-\frac{1}{x}\right) \to -\infty$$

   and

$$1-\frac{1}{x} \to -\infty$$

   hence by the squeeze theorem we find that

$$\frac{\log(x)}{x} \to -\infty\,.$$

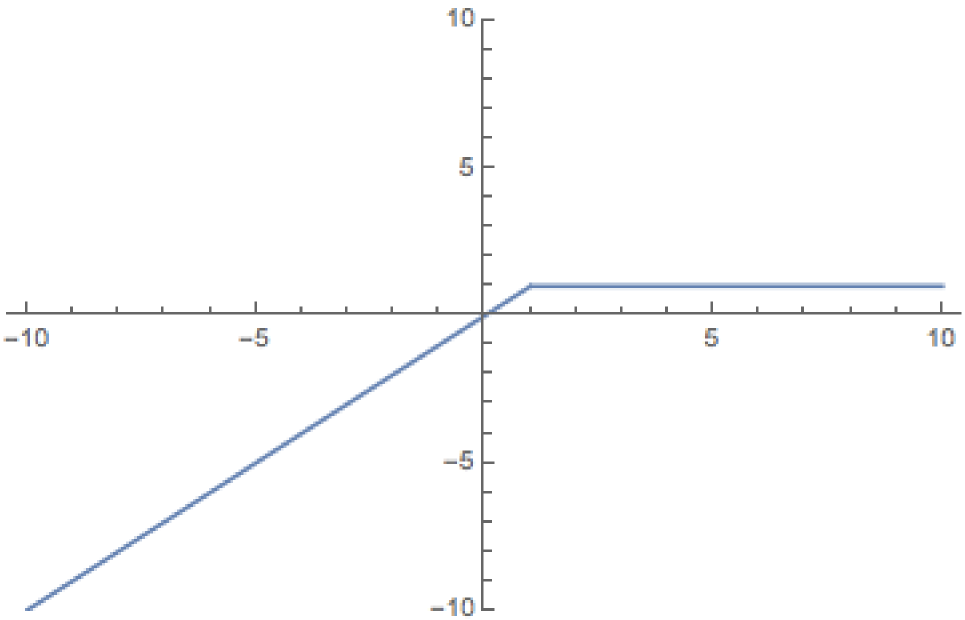
Exercise 14. Determine the *set* where the following functions are differentiable and find the derivative on that set. No proof is necessary.

1. $\mathbb{R}\ni x\mapsto x$.
   This is just the identity function which is differentiable everywhere, and its derivative is the constant function $x\mapsto 1$.

2. $\mathbb{R}\ni x\mapsto |x|$.
   As we saw in class (lecture notes Example 8.8), $x\mapsto|x|$ is differentiable on $\mathbb{R}\setminus\{0\}$, and its derivative is equal

$$(\mathbb{R}\setminus\{0\}\ni x\mapsto|x|)' = \begin{cases}1 & x>0\\ -1 & x<0\end{cases}\,.$$

3. $\mathbb{R}\ni x\mapsto\begin{cases}1 & x\geq 1\\ x & x<1\end{cases}$.
   The graph of this function looks like



and we can differentiate the two regions separately, from which we get $x \mapsto 1$ on $(-\infty, 1)$ and $x \mapsto 0$ on $(1, \infty)$. The question is whether the function (let's call it f) is differentiable at 1 (it is not) and the way to see this is to evaluate the limit

$$\begin{aligned} f'(1) &\stackrel{?}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(f(1+\varepsilon) - f(1) \right) \\ &\quad (\text{Use } f(1) = 1) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(f(1+\varepsilon) - 1 \right) \\ &= \begin{cases} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left(f(1+\varepsilon) - 1 \right) \\ \lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} \left(f(1+\varepsilon) - f(1) \right) \end{cases} \\ &= \begin{cases} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (1-1) \\ \lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} (1+\varepsilon-1) \end{cases} \\ &= \begin{cases} \lim_{\varepsilon \to 0^+} 0 \\ \lim_{\varepsilon \to 0^-} 1 \end{cases} \end{aligned}$$

Since the limits from either side of 1 are not equal, we conclude that the general limit does not exist, and hence, f is *not* differentiable at 1.

Exercise 15. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 5f(x)$ for all $x \in \mathbb{R}$ and such that $f(0) = 1$.

Solution. For us, the way to solve such problems is to *guess*. Once one has acquired familiarity with several examples (there are actually very few which can be explicitly solved) one tries to combine them together to find a solution to a particular problem at hand.

The point is we know that $\exp' = \exp$, and this is the basic building block in many such problems. So let us try a function similar to this with the chain rule:

$$\begin{aligned} (x \mapsto \exp(5x))' &= (x \mapsto \exp(5x))\,(x \mapsto 5x)' \\ &= (x \mapsto \exp(5x))\,5 \\ &= x \mapsto 5\exp(5x) \end{aligned}$$

In addition,

$$\begin{aligned} \exp(5 \cdot 0) &= \exp(0) \\ &= 1 \end{aligned}$$

So the function $f(x) := \exp(5x)$ for all $x \in \mathbb{R}$ does the job.