Calculus 1 – Spring 2019 Section 2–HW5

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1 Review

Exercise 1. Simplify the following expressions:

1. $x, y \in \mathbb{R}$:
$$\frac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}} \, .$$

2. $\alpha, \beta \in \mathbb{R}$:
$$\frac{\left(\alpha^4 \beta^{-3}\right)^2}{\alpha^3 \beta^5} \, ,$$
and
$$\frac{\frac{1}{\beta^2} + \alpha^2}{\frac{1}{\alpha^2} + \beta^2} \, .$$

Exercise 2. Solve for $x \in \mathbb{R}$:
$$\log_{10}(x-1) = 2 \, .$$

Exercise 3. Factorize or complete the square for the following expressions for $\varepsilon, \delta \in \mathbb{R}$:
$$\varepsilon^4 \delta - \delta^4 \varepsilon \, .$$

Exercise 4. For each of the following inequalities, provide an interval, open or closed, such that if x belongs to that interval, it satisfies the respective inequality:

1. $x(x-1)(x+2) > 0$.
2. $|x-3| < 4$.
3. $x^2 < 3x + 8$.
4. $\frac{2x-3}{x+1} \leq 1$.

Exercise 5. What equation is satisfied by all points on the plane $(x, y) \in \mathbb{R}^2$ which lie on the circle centered at the point $(a, b) \in \mathbb{R}^2$ and passing through the point $(c, d) \in \mathbb{R}^2$?

Exercise 6. Sketch the region of $\mathbb{R}^2$ corresponding to all points $(x, y) \in \mathbb{R}^2$ which satisfy the following equalities or inequalities in x, y:

1. $-2 \leq y < 4$.
2. $x^2 + y^2 > 9$.
3. $|x| < 2$ and $|y| < 8$.
4. $\frac{x^2}{2} + \frac{y^2}{3} < 4$.
5. $|x| + |y| = 1$ and $x > 0$.

6. $\sin(2x) = \sin(x)$ and $x \in [0, 2\pi]$.

Exercise 7. Sketch the graph of the functions, clearly marking where they pass through the horizontal or vertical axes (if they do), and what happens at $\pm\infty$ (when relevant):

1. $\log_2 : (0, \infty) \to \mathbb{R}$.
2. $\exp_2 : \mathbb{R} \to (0, \infty)$.
3. $\mathbb{R} \ni x \mapsto 1 + \sin(2x) \in \mathbb{R}$.
4. $(0, \infty) \ni x \mapsto \sqrt{x} \in \mathbb{R}$.
5. $(0, \infty) \ni x \mapsto 2\sqrt{x} \in \mathbb{R}$.
6. $\mathbb{R} \setminus \{ 0 \} \ni x \mapsto 1 + \frac{1}{x} \in \mathbb{R}$.

Exercise 8. What are all values of $x \in [-10, 10]$ where the following expressions are zero? (restrict x when necessary)

1. $\log_{\mathrm{e}}(x)$? (Recall $\mathrm{e} \approx 2.718$ from the appendix of the lecture notes)
2. $\sin(3x)$?
3. $\tan(x) + \sin(x)$?

Exercise 9. Solve the following limits (possibly diverging to $\pm\infty$), or write "does not exist". Note you do not need to prove the existence from the definition.

1. $\lim_{n\to\infty} (\lim_{m\to\infty} \alpha)$ for some $\alpha \in \mathbb{R}$.
2. $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} n\right)$. Note this is the limit of a *sequence*, i.e. $n \in \mathbb{N}$ here.
3. $\lim_{x\to-\infty} \exp_a(x)$ for $a > 1$.
4. $\lim_{m\to\infty} \left(\lim_{n\to\infty} \frac{m}{m+n}\right)$.
5. $\lim_{n\to\infty} \left(\lim_{m\to\infty} \frac{m}{m+n}\right)$.
6. $\lim_{n\to\infty} \sum_{k=0}^{n} ar^k$ for $a \in \mathbb{R}$ and $r < 1$. You may consult the appendix in the lecture notes if you find the symbol $\sum$ unfamiliar; use the formula $\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r}$.
7. $\lim_{n\to\infty} \sum_{k=0}^{n} ar^k$ for $a \in \mathbb{R}$ and $r > 1$.

2 Exercises pertaining to new material

Exercise 10. Calculate the derivative of the following functions (no proof necessary) and restrict the domain of the derivative if necessary:

1. $x \mapsto \frac{\sec(x)}{1+\tan(x)}$.
2. $x \mapsto (\cos(x))^2$, $x \mapsto \cos\left(x^2\right)$ and $x \mapsto \cos(\cos(x))$.
3. tan.
4. cot.
5. $\mathbb{R} \ni x \mapsto (\mathrm{e}^x)^2$.
6. $\mathbb{R} \ni x \mapsto \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}}$.
7. $\mathbb{R} \ni x \mapsto 5$.
8. $(0, \infty) \ni x \mapsto \sqrt[3]{x}$.

9. $\mathbb{R} \setminus \{ 0 \} \ni x \mapsto \frac{5x^2+10x+20}{80x^{100}}$.

10. $\mathbb{R} \setminus \{ 0 \} \ni x \mapsto \frac{1}{|x|}$ whenever possible.

11. $\exp \circ \cos$.

Exercise 11. If f is differentiable then f' is a new function, on which we may *yet again* ask whether f' itself differentiable. If it is, then we can calculate in turn its own derivative, $(f')'$ which is called *the second derivative* (denoted by f'') of f. Calculate the second derivative of the following functions:

1. $\mathbb{R} \ni x \mapsto x$.

2. $\mathbb{R} \ni x \mapsto x^2$.

3. $\mathbb{R} \setminus \{ 0 \} \ni x \mapsto \frac{1}{x}$.

Exercise 12. Determine whether the following functions are increasing or decreasing by examining the sign of their derivative. Recall that if the derivative was positive at some point, then the function was increasing and if it was negative the function was decreasing.

1. exp.

2. log.

3. $x \mapsto x^3$.

4. $(0, \infty) \ni x \mapsto \sqrt{x}$.

Exercise 13. Use the so-called *the hospital rule,* when appropriate (sometimes you can just proceed directly) in order to evaluate the following limits.

1. For any $n \in \mathbb{N}$, $\lim_{x\to\infty} \frac{x^n}{e^x}$.

2. $\lim_{x\to 0} x \log(x)$ (restrict the function to $x > 0$ for the logarithm to make sense).

3. $\lim_{x\to 3} \frac{\cos(x)\log(x-3)}{\log(e^x - e^3)}$.

4. $\lim_{x\to 0} \frac{e^x - 1}{\sin(x)}$.

5. $\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)}$ for two constants $p, q \in \mathbb{R}$.

6. $\lim_{x\to 0} \frac{\log(x)}{x}$ (restrict x as necessary to make the logarithm make sense).

Exercise 14. Determine the *set* where the following functions are differentiable and find the derivative on that set. No proof is necessary.

1. $\mathbb{R} \ni x \mapsto x$.

2. $\mathbb{R} \ni x \mapsto |x|$.

3. $\mathbb{R} \ni x \mapsto \begin{cases} 1 & x \geq 1 \\ x & x < 1 \end{cases}$.

Exercise 15. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 5f(x)$ for all $x \in \mathbb{R}$ and such that $f(0) = 1$.