Calculus 1 – Spring 2019 Section 2–HW6–Solutions

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1 Review

Exercise 1. Find an equation for the line that passes through the point $(2, 2)$ and is parallel to the vertical axis.

Solution. We use the variables $(x, y) \in \mathbb{R}^2$ to denote the horizontal (respectively vertical) coordinates of any point on the plane. Being parallel to the vertical axis means the equation must for the line must be: all points $(x, y) \in \mathbb{R}^2$ obeying the equation $x = a$. Here, $a \in \mathbb{R}$ is a free parameter that we must find and apparently y is unconstrained for vertical lines. When we put the constraint that the line must pass through the point $(x, y) = (2, 2)$ we get that $a = 2$, so that the equation is: all points $(x, y) \in \mathbb{R}^2$ such that

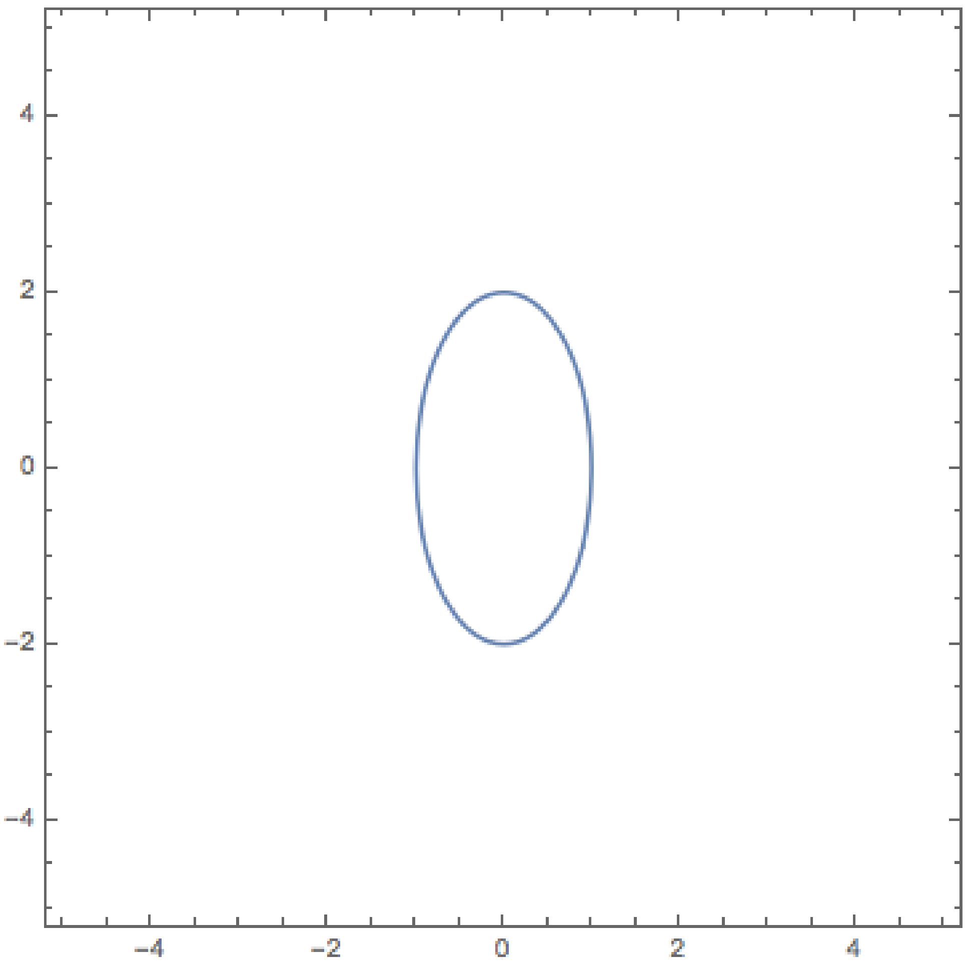
$$x = 2$$

Exercise 2. Draw the shape formed by all points $(x, y) \in \mathbb{R}^2$ obeying the equation

$$x^2 + \frac{y^2}{4} = 1$$

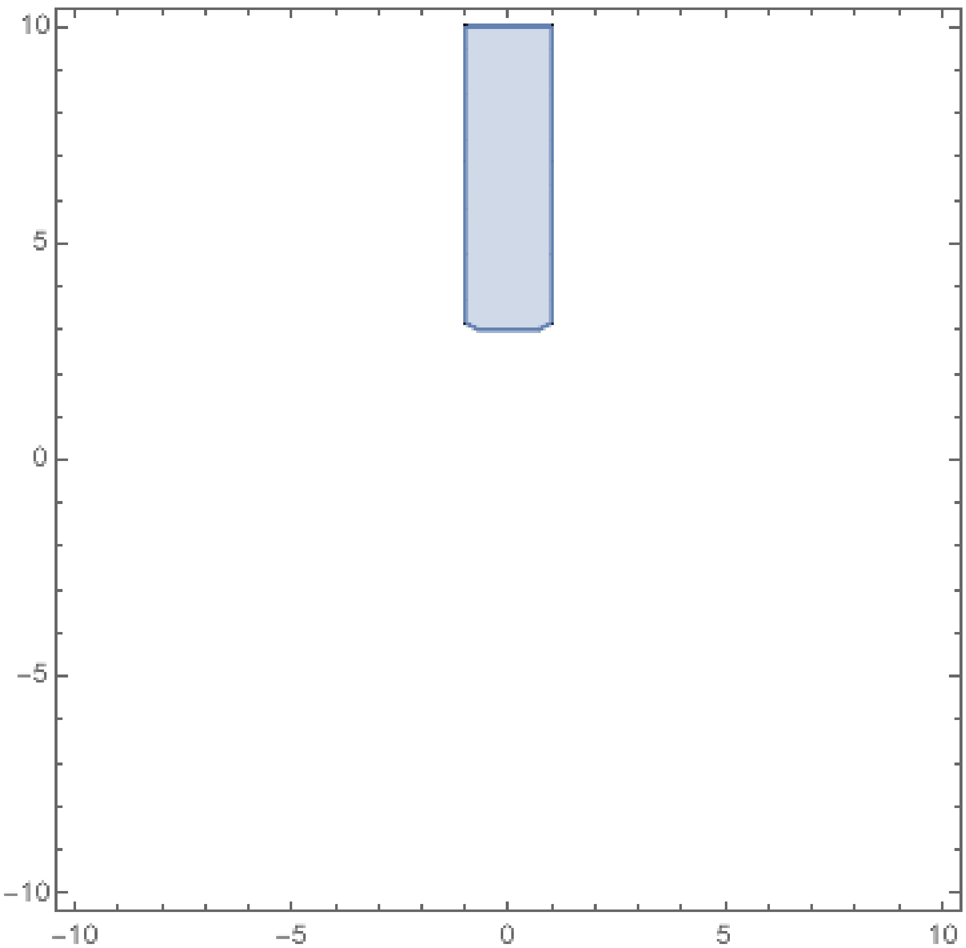
as accurately as you can.

Solution. We use Mathematica's *ContourPlot* to get



Exercise 3. Draw the region of $\mathbb{R}^2$ of all points $(x, y) \in \mathbb{R}^2$ obeying the constraint $|x| < 1$ and $y > 3$.

Solution. We have $-1 < x < 1$ and $y > 3$ which corresponds to a half-infinite rectangular strip extending upwards



Exercise 4. Express $\tan(2x)$ in terms of some arithmetic operations in $\tan(x)$.

Solution. We have

$$\begin{aligned}\tan(2x) &\equiv \frac{\sin(2x)}{\cos(2x)} \\ &\quad \text{(Use trig. id.)} \\ &= \frac{2\sin(x)\cos(x)}{\cos(x)^2-\sin(x)^2} \\ &\quad \left(\text{Divide by } \cos(x)^2\right) \\ &= \frac{2\tan(x)}{1-\tan(x)^2}\,.\end{aligned}$$

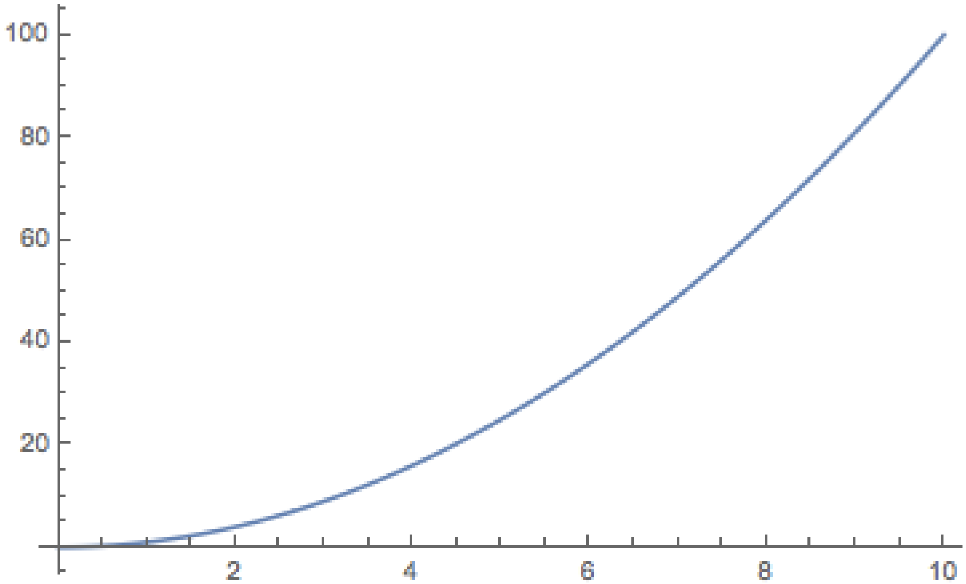
Exercise 5. Express $\cos(x+y)$ in terms of $\cos(x)$, $\sin(x)$, $\cos(y)$ and $\sin(y)$.

Solution. We have the trigonometric identity

$$\cos(x+y) = \cos(x)\cos(y)-\sin(x)\sin(y)\,.$$

Exercise 6. If the sketch of the graph of $f(x)=x^2$ restricted to positive x is rotated by $\frac{\pi}{2}$-radians clock-wise, we get the sketch of the graph of a function given by which formula?

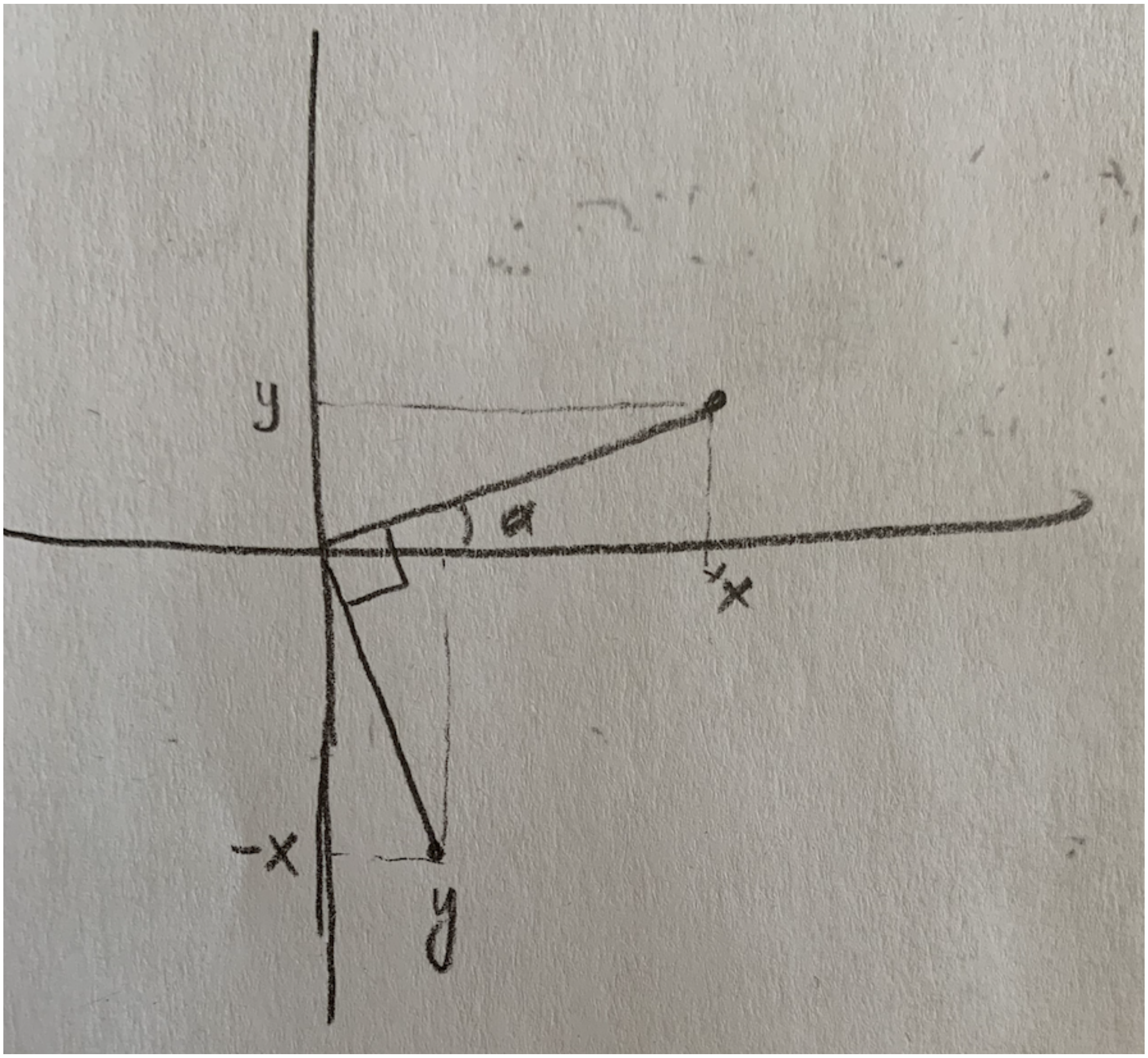
Solution. The graph of $(0,\infty)\ni x\mapsto x^2$ looks as follows



more generally, the graph of the function is the set $\operatorname{graph}(f) \equiv \{ (x,y) \in \mathbb{R}^2 \mid y = f(x) \}$ and here we have

$$\begin{aligned} \operatorname{graph}(f) &\equiv \{ (x,y) \in \mathbb{R}^2 \mid y = x^2 \wedge x > 0 \} \\ &= \{ (x, x^2) \in \mathbb{R}^2 \mid x > 0 \} \end{aligned}$$

to rotate a point on $\mathbb{R}^2$ by $\frac{\pi}{2}$ clockwise, we make the replacement $(x,y) \mapsto (y, -x)$, as in the following drawing:



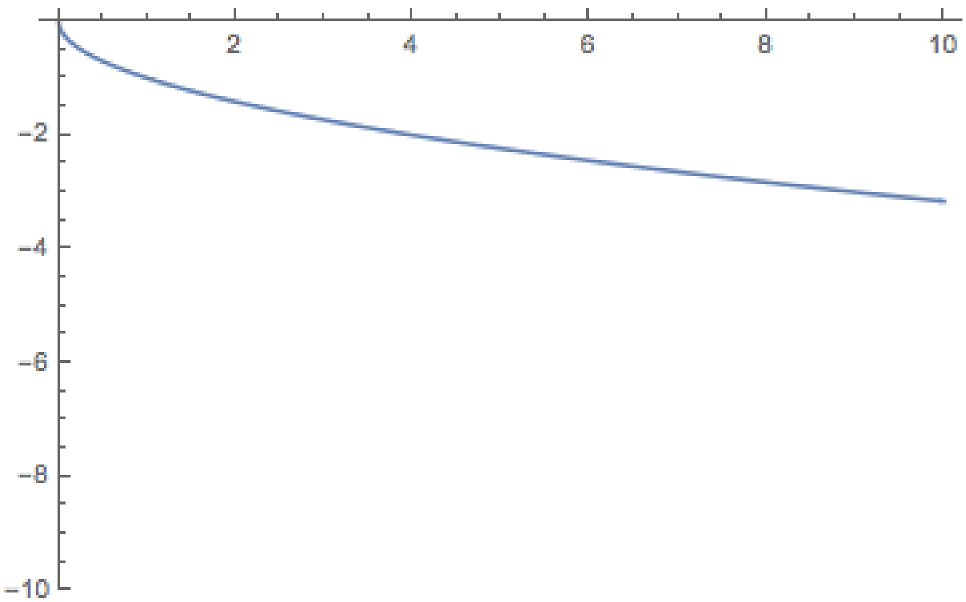
so we ask what is the function g whose graph is given by

$$\text{graph}\,(g) \quad = \quad \left\{\, \left(x^2, -x\right) \in \mathbb{R}^2 \;\middle|\; x > 0 \,\right\}$$

and now we'd like to have a formula for g. To get this formula, we need to bring the set to the form graph $(g) \equiv \left\{\, (x, y) \in \mathbb{R}^2 \;\middle|\; y = g\,(x) \,\right\}$, which is apparently the case if we use the following equivalence of sets:

$$\left\{\, \left(x^2, -x\right) \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \,\right\} \quad = \quad \left\{\, \left(t, -\sqrt{t}\right) \in \mathbb{R}^2 \;\middle|\; \sqrt{t} > 0 \,\right\}$$

so we find that if we define $g\,(x) := -\sqrt{x}$ for all $x > 0$ this makes sense, and indeed, the plot shows



2 Exercises pertaining to new material

Exercise 7. Calculate $\lim_{\varepsilon\to 0}\left(1+2\sin\left(\varepsilon\right)\right)^{\cot(\varepsilon)}$ using the hospital rule (eventually).

Solution. We have

$$\begin{aligned}\left(1+2\sin\left(\varepsilon\right)\right)^{\cot(\varepsilon)} &= \exp\left(\log\left(\left(1+2\sin\left(\varepsilon\right)\right)^{\cot(\varepsilon)}\right)\right)\\ &= \exp\left(\cot\left(\varepsilon\right)\log\left(\left(1+2\sin\left(\varepsilon\right)\right)\right)\right)\\ &= \exp\left(\cos\left(\varepsilon\right)\frac{\log\left(\left(1+2\sin\left(\varepsilon\right)\right)\right)}{\sin\left(\varepsilon\right)}\right)\end{aligned}$$

We note that as $\varepsilon\to 0$, $\cos\left(\varepsilon\right)\to 1$ which is fine, but the term $\frac{\log((1+2\sin(\varepsilon)))}{\sin(\varepsilon)}\to\frac{0}{0}$ which suggest to use l'Hospital's rule on it. Let us see what happens with the quotient of the derivatives:

$$\begin{aligned}\frac{\left(\log\left(\left(1+2\sin\left(\varepsilon\right)\right)\right)\right)'}{\left(\sin\left(\varepsilon\right)\right)'} &= \frac{\frac{1}{1+2\sin(\varepsilon)}2\cos\left(\varepsilon\right)}{\cos\left(\varepsilon\right)}\\ &= \frac{2}{1+2\sin\left(\varepsilon\right)}\\ &\to 2\end{aligned}$$

so apparently l'Hospital's rule may be applied and we learn that

$$\lim_{\varepsilon\to 0}\frac{\log\left(\left(1+2\sin\left(\varepsilon\right)\right)\right)}{\sin\left(\varepsilon\right)} = 2\,.$$

Since exp is continuous, we may push the limit through, and so all together the result of the limit is

$$\lim_{\varepsilon\to 0}\left(1+2\sin\left(\varepsilon\right)\right)^{\cot(\varepsilon)} = \mathrm{e}^2\,.$$

Exercise 8. Calculate f' if $f\left(x\right)=\sin\left(x\right)^2+\cos\left(x\right)^2$ for all $x\in\mathbb{R}$.

Solution. Using the trigonometric identity, we have $f\left(x\right)=1$ for all $x\in\mathbb{R}$, the constant function, whose derivative is the constant zero function, i.e. $f'\left(x\right)=0$ for all $x\in\mathbb{R}$.

Exercise 9. Determine whether the following functions is differentiable (and where) and whether its derivative is continuous (and if so where):

$$\begin{aligned} f:\mathbb{R} &\to \mathbb{R}\\ x &\mapsto \begin{cases} x^2\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}\end{aligned}$$

Solution. This has been presented in the lecture notes on Example 8.28.

Exercise 10. Let $f(x) := (x - \sqrt{x})^2$. Find f' and f''.

Solution. We have

$$\begin{aligned} f'(x) &= 2\left(x - \sqrt{x}\right)\left(1 - \frac{1}{2\sqrt{x}}\right) \\ &= 2\left(x - \sqrt{x}\right) - \sqrt{x} + 1 \\ &= 2x - 3\sqrt{x} + 1 \end{aligned}$$

now differentiating again we find

$$\begin{aligned} f''(x) &= 2 - 3\frac{1}{2\sqrt{x}} \\ &= 2 - \frac{3}{2}\frac{1}{\sqrt{x}}\,. \end{aligned}$$

Exercise 11. Find two (different) functions $f : \mathbb{R} \to \mathbb{R}$ which obey $f'' = \alpha^2 f$ and $f(0) = 1$ for some $\alpha \in \mathbb{R}$. Recall f'' is the *second* derivative of f, i.e., it is the derivative of f'. Also recall that $\exp' = \exp$ and use the chain rule.

Solution. If $f(x) := \exp(\alpha x)$ then $f'(x) = \alpha \exp(\alpha x) = \alpha f(x)$ and so $f''(x) = \alpha^2 \exp(\alpha x) = \alpha^2 f(x)$. Also note that $f(0) = \exp(\alpha \cdot 0) = \exp(0) = 1$. The point now is that if we replace α by $-\alpha$, the first derivative changes, but the second one doesn't, since we have $\alpha^2 \mapsto (-\alpha)^2 = \alpha^2$. Hence the two different functions are

$$x \quad \mapsto \quad \exp(\pm \alpha x)\,.$$

Exercise 12. Find a function f such that $f' = -f^2$ and such that $f(1) = 1$.

Solution. By guessing (i.e. having had exposure to the derivatives of several model functions and keeping them in mind), we note that if $f(x) = \frac{1}{x}$ for all $x \neq 0$ then $f'(x) = -\frac{1}{x^2} = -\left(\frac{1}{x}\right)^2 = -f(x)^2$. Also note that $f(1) = \frac{1}{1} = 1$.

Exercise 13. Let $f(x) := \left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^3$ for all x for which this makes sense (see the definition of tanh in the lecture notes). Calculate f'.

Solution. We have, using the chain rule

$$\begin{aligned}
f'(x) &= 3\left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^2\left(x\mapsto\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)' \\
&= 3\left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^2\left(\frac{1}{\frac{1}{\sqrt{\tanh(x)}}}\right)\left(x\mapsto\frac{1}{\sqrt{\tanh(x)}}\right)' \\
&= 3\left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^2\sqrt{\tanh(x)}\left(-\frac{1}{2}\right)(\tanh(x))^{-\frac{3}{2}}\tanh'(x) \\
&\quad \left(\text{Use } \tanh' = \frac{1}{\cosh^2} \text{ from lecture notes 8.30.}\right) \\
&= -\frac{3}{2}\left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^2\frac{1}{\tanh(x)}\frac{1}{\cosh(x)^2} \\
&\quad \left(\text{Use } \tanh \equiv \frac{\sinh}{\cosh}\right) \\
&= -\frac{3}{2}\left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^2\frac{1}{\sinh(x)\cosh(x)} \\
&\quad (\text{Use } 2\sinh\cosh = \sinh 2\cdot) \\
&= -3\left(\log\left(\frac{1}{\sqrt{\tanh(x)}}\right)\right)^2\frac{1}{\sinh(2x)} \\
&\quad (\text{Use logarithm laws}) \\
&= -3\left(\left(-\frac{1}{2}\right)\log(\tanh(x))\right)^2\frac{1}{\sinh(2x)} \\
&= -\frac{3}{4}(\log(\tanh(x)))^2\frac{1}{\sinh(2x)}\,.
\end{aligned}$$

Exercise 14. Find all points at which the *derivative* of the following functions is zero. Such points are called *critical points.* Say (if you can) whether these points which you find are (global or local) maxima or minima.

1. $f:\mathbb{R}\to\mathbb{R}$ given by $x\mapsto x^2$.
   The critical point is zero, and it is a global minimum.

2. $f:\mathbb{R}\to\mathbb{R}$ given by $x\mapsto\sinh(x)^2$.
   We have $f'(x) = 2\sinh(x)\sinh'(x) = 2\sinh(x)\cosh(x) = \sinh(2x) = \frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{2}$, which is zero when $\mathrm{e}^{2x} = \mathrm{e}^{-2x}$. Taking the logarithm that is when $2x=-2x$, or when $x=0$. Since $f'(x)<0$ when $x<0$ and $f'(x)>0$ when $x>0$ (to see this, check when

   $$\mathrm{e}^{2x} \quad > \quad \mathrm{e}^{-2x}$$

   which is whenever $x>0$), we see that zero is an inflection point for f, and it turns out that this is a global minimum, since the function is going down until $x=0$, and then going up starting from $x=0$.

3. $f:\mathbb{R}\to\mathbb{R}$ given by $x\mapsto\tanh'(x)$ (i.e. the *derivative* of tanh).
   Recall $\tanh' = \frac{1}{\cosh^2}$ from lecture notes 8.30. Now we want to inquire about the derivative of $\frac{1}{\cosh^2}$, which is

   $$\begin{aligned}
   \left(\frac{1}{\cosh^2}\right)' &= -2\cosh^{-3}\cosh' \\
   &= -2\cosh^{-3}\sinh \\
   &= -2\cosh^{-2}\tanh
   \end{aligned}$$

   Note that cosh is never zero: it grows very quickly to infinity as one takes its argument to $\pm\infty$, and it is 1 at zero. sinh on the other hand goes from $-\infty$ at $-\infty$ to $+\infty$ at $+\infty$ through zero at zero. Hence $-2\cosh^{-3}\sinh$ is only zero at zero and at $\pm\infty$: on the left of zero it's always positive and on the right of zero it's always negative. We learn that $\tanh''$ has one inflection point at zero, and otherwise goes from being positive to negative, hence $\tanh'$ has one global maximum at zero: it is increasing up to that point and then it starts decreasing.

Exercise 15. Do the hypothesis of the mean value theorem (see lecture notes) hold for $f(x) = 3x^2 - \frac{2}{x}$ on the interval $[-1, 1]$?

Solution. The hypothesis of the MVT do not hold for this function on $[-1, 1]$: it is not defined at zero, so it is not continuous there, and clearly not differentiable there.

Exercise 16. Johannes is driving on I95 which has a speed limit of $55mph$ southwards from New York City. At 10am he was 110 miles away from NYC and at 1pm he was 290 miles away from NYC. Provide evidence to the court that Johannes was speeding, by assuming that the function f : time $\to$ miles away from NYC is a continuous function and differentiable, and its derivative is precisely the *instantaneous speed* of Johannes.

Solution. Let $f : [0, \infty) \to \mathbb{R}$ denote the distance (in miles) of Johannes from NYC as a function of time (in hours). We assume that f is differentiable (hence continuous). We parameterize time such that at that day, at $10am$, we set the zero mark. Since we are given that at that time point Johannes was 110 miles away from NYC, we know that

$$f(0) = 110 .$$

We are also given that three hours later is he is 290 miles away, so

$$f(3) = 290 .$$

We apply the MVT on f on the interval $[0, 3]$, which now says that there must be some $t \in (0, 3)$ such that

$$\begin{aligned} f(t) &= \frac{f(3) - f(0)}{3 - 0} \\ &= \frac{290 - 110}{3} \\ &= \frac{180}{3} \\ &= 60 . \end{aligned}$$

i.e., there *must* have been a time point between 10am and 1pm in which Johannes' speed was 60 mph, which is 5 mph above the speed limit.