Calculus 1 – Spring 2019 Section 2
HW7 *Solutions*

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April 3, 2019

Remark. The due date was April 3rd, 2019.

1 Review

1.1 Exercise. Apply the following functions on the number $x \in \mathbb{R}$ (or a subset of $\mathbb{R}$ if need be to make sense). E.g. if the function is $\sin \circ \log$, your answer should be $\sin(\log(x))$.

1. $\cosh \circ \sin \circ \exp \circ \log$.
2. $\left(\frac{\sin}{\cos}\right)^2$.
3. $\sin \circ \sin$.
4. $\sin \sin$.
5. $\frac{1}{\cos} \circ \arccos$.

Solution. We have

1. $x \mapsto \cosh(\sin(x))$ (as $\exp \circ \log = \mathbb{1}$) as long as $x > 0$.
2. $x \mapsto \frac{\sin(x)^2}{\cos(x)^2}$.
3. $x \mapsto \sin(\sin(x))$.
4. $x \mapsto \sin(x)\sin(x)$.
5. $x \mapsto \frac{1}{x}$ (as $\cos \circ \arccos = \mathbb{1}$) as long as $x \neq 0$.

1.2 Exercise. Find the slope of the line in $\mathbb{R}^2$ passing through $(5,6) \in \mathbb{R}^2$ and tangent to the circle centered at $(0,0) \in \mathbb{R}^2$ with radius 2.

Solution. A straight line must have the equation

$$f(x) = \alpha x + \beta$$

for some $\alpha, \beta \in \mathbb{R}$ and α is the slope. Since the line passes through $(5,6)$, we have the equation

$$6 = 5\alpha + \beta .$$

From this we conclude $\beta = 6 - 5\alpha$ and so

$$\begin{aligned} f(x) &= \alpha x + 6 - 5\alpha \\ &= \alpha(x-5) + 6 . \end{aligned}$$

The circle centered at $(0,0) \in \mathbb{R}^2$ with radius 2 is the set of points $(x,y) \in \mathbb{R}^2$ obeying the equation $x^2 + y^2 = 4$. If we want f to be tangent to the circle, that means it should intersect with it at exactly one point, i.e., there should be only one solution to the equation (with unknown $x \in \mathbb{R}$)

$$x^2 + (f(x))^2 = 4$$

i.e.

$$
\begin{aligned}
x^2 + (\alpha (x-5) + 6)^2 &= 4 \\
x^2 + \alpha^2 (x-5)^2 + 2\alpha (x-5) 6 + 36 &= 4 \\
x^2 + \alpha^2 \left(x^2 - 10x + 25\right) + 12\alpha x - 60\alpha + 32 &= 0 \\
\left(1+\alpha^2\right) x^2 - 2 (5\alpha - 6) \alpha x + 25\alpha^2 - 60\alpha + 32 &= 0
\end{aligned}
$$

This is a quadratic equation in x; its discriminant is equal to

$$\sqrt{(2(5\alpha-6)\alpha)^2 - 4(1+\alpha^2)(25\alpha^2 - 60\alpha + 32)}$$

which better be equal to zero so that we indeed get only one solution for x. I.e. we find an equation for α given by

$$
\begin{aligned}
(2(5\alpha-6)\alpha)^2 - 4\left(1+\alpha^2\right)\left(25\alpha^2 - 60\alpha + 32\right) &\overset{!}{=} 0 \\
&\updownarrow \\
-4(32 + 3\alpha(-20 + 7\alpha)) &= 0
\end{aligned}
$$

which apparently has solution

$$\alpha = \frac{2}{21}\left(15 \pm \sqrt{57}\right) .$$

We find that there are two possible slopes for this straight line.

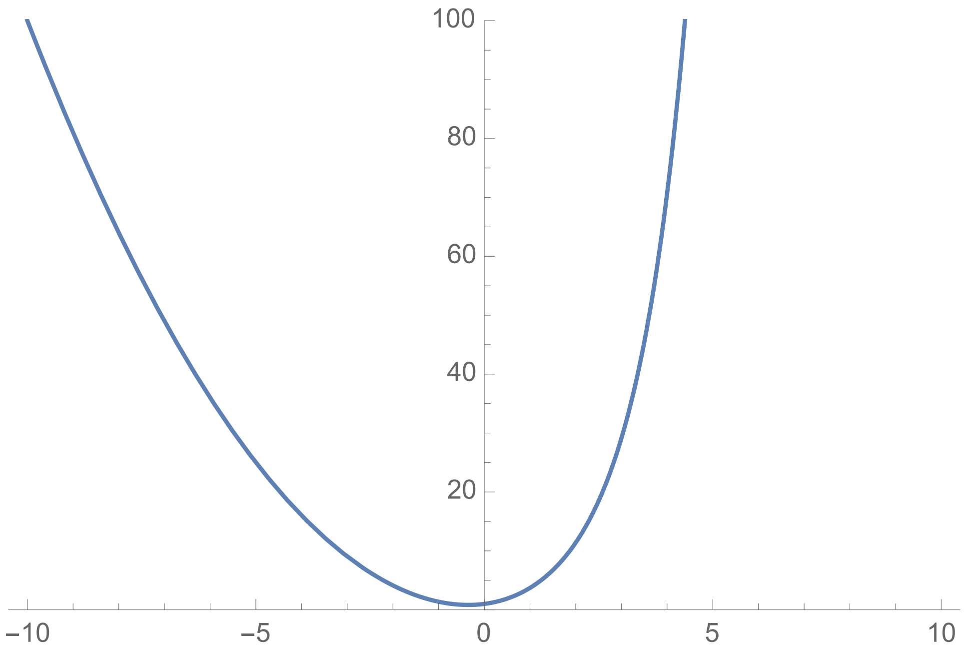
1.3 Exercise. Make a sketch of the graph of $\mathbb{R} \ni x \mapsto x^2 + \exp(x) \in \mathbb{R}$ by following the steps below:

1. Find out where it's positive and where it's negative.
2. Find out where it passes the horizontal axis.
3. Find out what happens at $\pm\infty$.
4. Study the derivative to find out where it is:
   (a) increasing / decreasing.
   (b) attains a global maximum / minimum.

Solution. The function is

1. Always positive.
2. Never crosses the horizontal axis.
3. Goes to ∞ at $\pm\infty$.
4. Its derivative is $x \mapsto 2x + \exp(x)$, which is going to be negative for large negative x and positive for positive positive x. Hence the functions attains a global minimum at the solution to $2x + \exp(x) = 0$ which is $\exp(x) = -2x$ (there is no explicit solution to this equation).

The graph's sketch looks like



1.4 Exercise. [Koenigsberger] Calculate the *sequential* limit

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt[n]{n}-1\right) .$$

Solution. We have

$$\begin{array}{rcl} \sqrt[n]{n} & = & \exp\left(\log\left(\sqrt[n]{n}\right)\right) \\ & = & \exp\left(\frac{1}{n}\log\left(n\right)\right) \\ & & (\log \text{ grows much slowlier than the linear function}) \\ & \longrightarrow & \exp\left(0\right) \\ & = & 1 \end{array}$$

yet $\sqrt{n}\to\infty$, so this limit is of the form $\infty\cdot 0$, hence a bit complicated, and to honestly solve this limit we must understand how much faster $\sqrt[n]{n}-1\to 0$ compared to $\sqrt{n}\to\infty$.

To that end, note the trivial identity

$$\left(\sqrt[n]{n}-1+1\right)^n \quad = \quad n$$

and also the fact that $\sqrt[n]{n}-1\geq 0$. Indeed, we have the equivalent conditions.

$$\begin{array}{rcl} \sqrt[n]{n}-1 & \geq & 0 \\ & \updownarrow & \\ \sqrt[n]{n} & \geq & 1 \\ & \updownarrow & \\ n & \geq & 1\,. \end{array}$$

Hence we may re-write, for all $n \geq 4$, the following chain of estimates (this is a bit creative...)

$$\begin{aligned}
\left(\sqrt[n]{n}-1+1\right)^n &= \sum_{j=0}^{n}\binom{n}{j}\left(\sqrt[n]{n}-1\right)^j \\
&\quad \text{(All terms are positive, so take only the } j=3 \text{ term to estimate from below)} \\
&\geq \binom{n}{3}\left(\sqrt[n]{n}-1\right)^3 \\
&= \frac{n(n-1)(n-2)}{6}\left(\sqrt[n]{n}-1\right)^3 \\
&\quad \left((n-1)(n-2)\geq n(n-3) \text{ as you can verify.}\right) \\
&\geq \frac{n\cdot n(n-3)}{6}\left(\sqrt[n]{n}-1\right)^3 \\
&= \frac{n^3}{6}\frac{n-3}{n}\left(\sqrt[n]{n}-1\right)^3 \\
&\quad \left(\frac{n-3}{n}\geq\frac{1}{4} \text{ for } n\geq 4\,.\right) \\
&\geq \frac{n^3}{6}\frac{1}{4}\left(\sqrt[n]{n}-1\right)^3 \\
&= \frac{n^3}{24}\left(\sqrt[n]{n}-1\right)^3
\end{aligned}$$

from which we learn $\sqrt[n]{n}-1\leq\sqrt[3]{24}n^{-\frac{2}{3}}$. After multiplying by $\sqrt{n}$ we get

$$\begin{aligned}
\sqrt{n}\left(\sqrt[n]{n}-1\right) &\leq \sqrt{n}\left(\sqrt[3]{24}n^{-\frac{2}{3}}\right) \\
&= \sqrt[3]{24}n^{\frac{1}{2}}n^{-\frac{2}{3}} \\
&= \sqrt[3]{24}n^{\frac{3}{6}-\frac{4}{6}} \\
&= \sqrt[3]{24}n^{-\frac{1}{6}}\,.
\end{aligned}$$

Of course, we also have for large n that $\sqrt{n}\left(\sqrt[n]{n}-1\right)\geq 0$ so that we find

$$0\leq\ \sqrt{n}\left(\sqrt[n]{n}-1\right)\leq\ \sqrt[3]{24}n^{-\frac{1}{6}}\,.$$

Now we may use the squeeze theorem, using the fact that $\lim_{n\to\infty}n^{-\frac{1}{6}}=0$, and the left hand bound is the constant sequence zero. Since both sequences converge to zero, the middle sequence must converge to zero as well.

1.5 Exercise. [Koenigsberger] Define $\alpha := 10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}$ (a really big number). Define the following three sequences $a, b, c : \mathbb{N}\to\mathbb{R}$: For any $n\in\mathbb{N}$, they are given by the formulæ

$$\begin{aligned}
a(n) &:= \sqrt{n+\alpha}-\sqrt{n} \\
b(n) &:= \sqrt{n+\sqrt{n}}-\sqrt{n} \\
c(n) &:= \sqrt{n+\frac{n}{\alpha}}-\sqrt{n}
\end{aligned}$$

1. Show that for all $n\in\mathbb{N}$ such that $n<\alpha^2$,
$$a(n)>b(n)>c(n)\,.$$
2. Show that
$$\lim a\ =\ 0\,.$$
3. Show that
$$\lim b\ =\ \frac{1}{2}\,.$$
4. Show that
$$\lim c\ =\ \infty\,.$$

Solution. We follow the steps suggested:

1. Assume that $n < \alpha^2$. Then to show $a(n) > b(n)$, we have to show

$$\begin{aligned} \sqrt{n+\alpha}-\sqrt{n} &> \sqrt{n+\sqrt{n}}-\sqrt{n} \\ &\updownarrow \\ \sqrt{n+\alpha} &> \sqrt{n+\sqrt{n}} \\ &\updownarrow \\ n+\alpha &> n+\sqrt{n} \\ &\updownarrow \\ \alpha &> \sqrt{n} \\ &\updownarrow \\ \alpha^2 &> n\,. \end{aligned}$$

To show $b(n) > c(n)$, we must show

$$\begin{aligned} \sqrt{n+\sqrt{n}}-\sqrt{n} &> \sqrt{n+\frac{n}{\alpha}}-\sqrt{n} \\ &\updownarrow \\ n+\sqrt{n} &> n+\frac{n}{\alpha} \\ &\updownarrow \\ \sqrt{n} &> \frac{n}{\alpha} \\ &\updownarrow \\ \frac{1}{\sqrt{n}} &> \frac{1}{\alpha} \\ &\updownarrow \\ \frac{1}{n} &> \alpha^{-2} \\ &\updownarrow \\ n &< \alpha^2\,. \end{aligned}$$

Both of these inequalities are hence equivalent to our initial hypothesis.

2. We want $\lim_{n\to\infty} \sqrt{n+\alpha}-\sqrt{n} = 0$. Recall that $\sqrt{x}-\sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$ so that

$$\begin{aligned} \sqrt{n+\alpha}-\sqrt{n} &= \frac{n+\alpha-n}{\sqrt{n+\alpha}+\sqrt{n}} \\ &= \frac{\alpha}{\sqrt{n+\alpha}+\sqrt{n}}\,. \end{aligned}$$

Now taking the limit we find

$$\lim_{n\to\infty} \sqrt{n+\alpha}-\sqrt{n} = \alpha \lim_{n\to\infty} \frac{1}{\sqrt{n+\alpha}+\sqrt{n}}\,.$$

Since $\sqrt{n}\to\infty$ and $\sqrt{n+\alpha}\to\infty$, we find our result as $\frac{1}{\infty} = 0$.

3. Using the same factorization we have

$$\begin{aligned}\sqrt{n+\sqrt{n}}-\sqrt{n} &= \frac{n+\sqrt{n}-n}{\sqrt{n+\sqrt{n}}+\sqrt{n}} \\ &= \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} \\ &= \frac{1}{\frac{\sqrt{n+\sqrt{n}}}{\sqrt{n}}+1} \\ &= \frac{1}{\sqrt{1+\frac{\sqrt{n}}{n}}+1} \\ &= \frac{1}{\sqrt{1+n^{-\frac{1}{2}}}+1}\,.\end{aligned}$$

Now using the fact that $n^{-\frac{1}{2}} \to 0$ and the continuity of all other functions involved (so that we can push the limit through) we find the result.

4. Here, we have

$$\begin{aligned}\sqrt{n+\frac{n}{\alpha}}-\sqrt{n} &= \frac{n+\frac{n}{\alpha}-n}{\sqrt{n+\frac{n}{\alpha}}+\sqrt{n}} \\ &= \frac{\frac{n}{\alpha}}{\sqrt{n+\frac{n}{\alpha}}+\sqrt{n}} \\ &= \frac{\sqrt{n}}{\alpha\sqrt{1+\frac{1}{\alpha}}+\alpha}\,.\end{aligned}$$

Now the denominator is constant (in n) and the numerator grows to ∞.

2 Ongoing lecture material

2.1 Exercise. In this exercise we study the *linear* approximation of sin near zero:

1. Find the value of sin at zero, and call it $\alpha \in \mathbb{R}$.
2. Find the value of the derivative of sin at zero, and call it $\beta \in \mathbb{R}$.
3. Define a *linear* function (i.e. a function whose graph's sketch looks like a straight line) $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) \;:=\; \alpha + \beta x$$

4. Use the limit definition of the derivative

$$\sin'(0) \;\equiv\; \lim_{a\to 0} \frac{1}{a}\left(\sin(a) - \sin(0)\right)$$

in order to determine that for any $\varepsilon > 0$ there is some threshold $\delta(\varepsilon) > 0$ such that if $|x| < \delta(\varepsilon)$ then sin is arbitrarily close to f in the following way:

$$\alpha + \beta x - \varepsilon|x| < \quad \sin(x) \quad < \alpha + \beta x + \varepsilon|x|\,.$$

Conclude that since ε itself can be made arbitrarily small, the term extra correction term $\varepsilon|x|$ can be made arbitrarily small compared to βx (i.e. $\frac{\varepsilon|x|}{\beta x}$ can be made arbitrarily close to zero), as long as $|x| < \delta(\varepsilon)$, so that it is in this sense that f approximates sin.
You may consult the lecture notes at Remark 8.5.

Solution. Let's follow the steps:

1. We have $\sin(0) = 0 =: \alpha$.
2. ...and $\sin' = \cos$ so $\sin'(0) = \cos(0) = 1 =: \beta$.

3. We define then $f(x) := x$ for all $x \in \mathbb{R}$. This (simplest) straight line is to be the linear approximation of sin for "small" values of the argument.

4. The definition of the limit $\lim_{a\to 0} \frac{1}{a}(\sin(a) - \sin(0)) = \sin'(0) = 1$ literally means: For any $\varepsilon > 0$ there is some $\delta_\varepsilon > 0$ such that if $a \in \mathbb{R}$ is chosen such that $0 < |a| < \delta_\varepsilon$ then

$$\left|\frac{1}{a}\sin(a) - 1\right| < \varepsilon .$$

(We have already used that $\sin(0) = 0$ and $\sin'(0) = 1$). Since $a \neq 0$, let us multiply this inequality by $|a|$ to get

$$\begin{array}{ccc} |\sin(a) - a| & < & |a|\varepsilon \\ & \updownarrow & \\ a - |a|\varepsilon < & \sin(a) & < a + |a|\varepsilon . \end{array}$$

Which is our ultimate goal: For $a \in \mathbb{R}$ such that $|a|$ is small (i.e. $|a| < \delta_\varepsilon$), we can replace the complicated $\sin(a)$ with the simple a, up to an error term $|a|\varepsilon$, which is itself very small. Indeed, $|a|\varepsilon$ is arbitrarily small compared to a:

$$\begin{aligned} \left|\frac{\text{error term}}{\text{linear term}}\right| &= \left|\frac{|a|\varepsilon}{a}\right| \\ &= \varepsilon \end{aligned}$$

i.e. we can make the error term arbitrarily small compared to the linear term, since we are free to choose $\varepsilon > 0$ as small (yet strictly positive) as we like.

2.2 Exercise. Prove that the function $x^3 + x - 1 = 0$ for the unknown $x \in \mathbb{R}$ has precisely one real root using Rolle's theorem.

Solution. Define $f(x) := x^3 + x - 1$ for all $x \in \mathbb{R}$, this function being continuous and differentiable. Let us plug in a few numbers (via guessing) to find that

$$\begin{aligned} f(0) &= -1 \\ f(1) &= 1 + 1 - 1 = 1 . \end{aligned}$$

Now the intermediate value theorem (Theorem 7.10 in the lecture notes) implies that there is *at least one* $x \in [0, 1]$ such that $f(x) = 0$, as $-1 < 0 < 1$. Note that for negative x, $f(x)$ is always negative, so that there will be no solutions for $x < 0$. Also, for $x > 1$, $f(x) = x^3 + x - 1 > x - 1 > 0$, so there will be no solutions for $x > 1$.

Assume there is another solution in $[0, 1]$. I.e., assume there is some $y \in [0, 1] \setminus \{ x \}$ such that $f(y) = 0$. Then by Rolle's theorem (Theorem 8.43) there is some $z \in (x, y)$ or $z \in (y, x)$ (depending on whether $x < y$ or $y < x$–both are possible) such that $f'(z) = 0$. However,

$$\begin{aligned} f'(q) &= 3q^2 + 1 \\ &> 0 \text{ for all } q \in \mathbb{R} . \end{aligned}$$

I.e. we reach a contradiction.

2.3 Exercise. A metal oil container is to be manufactured so that it could contain volume (measured in cubic meters) $v \in \mathbb{R}$ of oil. The cost of fabricating the container is proportional to the surface area of the container, so that minimizing the surface area of the container minimizes the cost of fabrication.

1. Assuming one wants a cylindrically-shaped container, find the optimal radius $r > 0$ and height $h > 0$ of the cylinder such that the cost to fabricate a container of volume v is minimal. You may use the fact that the volume of a cylinder is

$$V_{\text{cylinder}} = \pi r^2 h$$

and its surface area is

$$A_{\text{cylinder}} = 2\pi r h + 2 \cdot \pi r^2 .$$

2. Assuming one wants a conus-shaped container, find the optimal radius $r > 0$ and height $h > 0$ of the conus such that the cost to fabricate a container of volume v is minimal. You may use the fact that the volume of a conus is

$$V_{\text{conus}} = \frac{1}{3}\pi r^2 h$$

and its surface area is

$$A_{\text{conus}} = \pi r^2 + \pi r\sqrt{r^2 + h^2}\,.$$

Hint: In either shape, plug in $V_{\text{shape}} = v$ to solve for h. Then plug this h into A_{shape} to obtain a function of r alone (it will also depend on v, but v is fixed throughout). Now find the derivative of

$$(0,\infty) \ni r \mapsto A_{\text{shape}} \in (0,\infty)$$

and equate it to zero to get an equation for r_{optimal}. After having found r_{optimal}, go back and find from this h_{optimal} in terms of r_{optimal} and v.

Solution. Let us deal with the two cases separately:

1. If we pick a cylinder shape, then we have

$$V = \pi r^2 h \overset{!}{=} v$$

from which we find $h = \frac{v}{\pi r^2}$. Hence the area as a function of the radius (the volume v is fixed) is

$$\begin{aligned} A(r) &= 2\pi r h + 2\cdot\pi r^2 \\ &= 2\pi r\frac{v}{\pi r^2} + 2\cdot\pi r^2 \\ &= 2\left(\pi r^2 + \frac{v}{r}\right). \end{aligned}$$

We differentiate this to find

$$A'(r) = 2\left(2\pi r - \frac{v}{r^2}\right).$$

We look for an extremum of $r \mapsto A(r)$ by solving $A'(r) = 0$ for r. We find

$$\begin{aligned} 2\pi r - \frac{v}{r^2} &= 0 \\ &\updownarrow \\ 2\pi r &= \frac{v}{r^2} \\ &\updownarrow \\ r^3 &= \frac{v}{2\pi} \\ &\updownarrow \\ r &= \sqrt[3]{\frac{v}{2\pi}}\,. \end{aligned}$$

This also shows that for $r < \sqrt[3]{\frac{v}{2\pi}}$ we have $A'(r) < 0$ (so in that range, $r \mapsto A(r)$ is decreasing) and for $r > \sqrt[3]{\frac{v}{2\pi}}$ we have $A'(r) > 0$ (so that $r \mapsto A(r)$ is increasing). These two facts together imply that $\sqrt[3]{\frac{v}{2\pi}}$ is an absolute minimum for $r \mapsto A(r)$.

2. For the conus, have follow the same procedure to get

$$\begin{aligned} \frac{1}{3}\pi r^2 h &\overset{!}{=} v \\ h &= \frac{v}{\frac{1}{3}\pi r^2} \end{aligned}$$

from which we find the area in terms of the radius alone (without the height)

$$\begin{aligned} A(r) &= \pi r^2 + \pi r\sqrt{r^2 + h^2} \\ &= \pi r^2 + \pi r\sqrt{r^2 + \left(\frac{v}{\frac{1}{3}\pi r^2}\right)^2} \\ &= \pi r^2 + \pi r\sqrt{r^2 + \frac{9v^2}{\pi^2 r^4}}\,. \end{aligned}$$

Differentiating this gives

$$\begin{aligned}
A'(r) &= 2\pi r + \pi\sqrt{r^2 + \frac{9v^2}{\pi^2 r^4}} + \pi r \frac{\left(2r - 4\frac{9v^2}{\pi^2} r^{-5}\right)}{2\sqrt{r^2 + \frac{9v^2}{\pi^2 r^4}}} \\
&= \frac{-9v^2 + 2\pi r^5\left(\pi r + \sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}}\right)}{r^4\sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}}} .
\end{aligned}$$

Note that $\pi^2 r^2 + \frac{9v^2}{r^4} \neq 0$ since both terms are always strictly positive separately. Anyway, we now again look for an extremal point by solving $A'(r) = 0$ for r:

$$\begin{aligned}
\frac{-9v^2 + 2\pi r^5\left(\pi r + \sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}}\right)}{r^4\sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}}} &\overset{!}{=} 0 \\
-9v^2 + 2\pi r^5\left(\pi r + \sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}}\right) &= 0 \\
\left(\pi r + \sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}}\right) &= 9v^2 \\
\sqrt{\pi^2 r^2 + \frac{9v^2}{r^4}} &= \frac{9v^2}{2\pi r^5} - \pi r \\
\pi^2 r^2 + \frac{9v^2}{r^4} &= \left(\frac{9v^2}{2\pi r^5} - \pi r\right)^2 \\
\pi^2 r^2 + \frac{9v^2}{r^4} &= \frac{81v^4}{4\pi^2 r^{10}} - \frac{9v^2}{r^4} + \pi^2 r^2 \\
\frac{9v^2}{r^4} &= \frac{81v^4}{4\pi^2 r^{10}} - \frac{9v^2}{r^4} \\
\frac{18v^2}{r^4} &= \frac{81v^4}{4\pi^2 r^{10}} \\
r^6 &= \frac{9v^2}{8\pi^2} \\
r &= \sqrt[6]{\frac{9v^2}{8\pi^2}} \\
&= \frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}} .
\end{aligned}$$

And again the same logic shows that A' is negative below $\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}}$ and positive above $\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}}$, i.e. A is decreasing below that radius and increasing above it, so that this radius is absolute minimum for A.

3. Finally we want to judge which shape is more fiscally favorable for the same given volume v. We have

$$\begin{aligned}
A_{\text{cylinder}}(r_{\text{optimal}}) &= 2\pi\sqrt[3]{\frac{v}{2\pi}}^2 + 2\frac{v}{\sqrt[3]{\frac{v}{2\pi}}} \\
&= 2\pi\left(\frac{v}{2\pi}\right)^{\frac{2}{3}} + 2(2\pi)^{\frac{1}{3}} v^{1-\frac{1}{3}} \\
&= \left((2\pi)^{\frac{1}{3}} + 2(2\pi)^{\frac{1}{3}}\right) v^{\frac{2}{3}} \\
&= 3(2\pi)^{\frac{1}{3}} v^{\frac{2}{3}} .
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
A_{\text{conus}}(r_{\text{optimal}}) &= \pi\left(\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}}\right)^2 + \pi\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}}\sqrt{\left(\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}}\right)^2 + \frac{9v^2}{\pi^2\left(\frac{1}{\sqrt{2}}\sqrt[3]{\frac{3v}{\pi}}\right)^2}} \\
&= 2 \times 3^{\frac{2}{3}}\pi^{\frac{1}{3}} v^{\frac{2}{3}} .
\end{aligned}$$

Now we want to compare these two to see which is bigger:

$$\begin{aligned}\frac{A_{\text{cylinder}}\left(r_{\text{optimal}}\right)}{A_{\text{conus}}\left(r_{\text{optimal}}\right)} &= \frac{3\left(2\pi\right)^{\frac{1}{3}}v^{\frac{2}{3}}}{2\times 3^{\frac{2}{3}}\pi^{\frac{1}{3}}v^{\frac{2}{3}}} \\ &= \frac{3\times 2^{\frac{1}{3}}}{2\times 3^{\frac{2}{3}}} \\ &= 2^{-\frac{2}{3}}3^{\frac{1}{3}} \\ &\approx 0.90 \\ &< 1\,.\end{aligned}$$

We learn that the conus is actually better suited than the cylinder!

2.4 Exercise. Find the point $(a,b)\in\mathbb{R}^2$ on the parabola defined by the set of all points $(x,y)\in\mathbb{R}^2$ obeying the equation

$$y^2 = 2x$$

such that the distance between (a,b) and $(1,4)\in\mathbb{R}^2$ is minimal.

Solution. We have to have $b^2 = 2a$ for (a,b) to be on the parabola. The distance between two points on the plane is given by the Pythagoras theorem as

$$\sqrt{\left(a-1\right)^2+\left(b-4\right)^2}$$

from the relation that (a,b) must be on the parabola we can express this in terms of b alone to find this distance, as a function of b, is

$$b \mapsto \sqrt{\left(\frac{1}{2}b^2-1\right)^2+\left(b-4\right)^2}\,.$$

Note that minimizing a function, we minimize its square root as well, since the square root is monotone increasing. Thus we could also work with

$$b \mapsto \left(\frac{1}{2}b^2-1\right)^2+\left(b-4\right)^2$$

for simplicity. Taking the derivative of this we find the function

$$\begin{aligned} b &\mapsto 2\left(\frac{1}{2}b^2-1\right)b+2\left(b-4\right) \\ &= b^3-2b+2b-8 \\ &= b^3-8\,.\end{aligned}$$

To find the extremal point we solve $b^3-8=0$ for b to find $b=2$. Hence $a=\frac{1}{2}b^2=\frac{1}{2}4=2$ and the point is $(2,2)\in\mathbb{R}^2$.

Note that the derivative is $b\mapsto b^3-8$ is negative for $b<2$ and positive for $b>2$ so that that the distance function is really at a minimum for this extremal point.

2.5 Exercise. At which points on the sketch of the function $\mathbb{R}\ni x\mapsto 1+40x^3-3x^5\in\mathbb{R}$ does the tangent line have the largest slope?

Solution. The slope of the tangent line to the function is given by the derivative,

$$x \mapsto 120x^2-15x^4\,.$$

We want that slope, in turn, to be extremal, so we differentiate once more and equate to zero:

$$\begin{aligned} 240x-60x^3 &\overset{!}{=} 0 \\ 4-x^2 &= 0 \\ x &= \pm 2\,.\end{aligned}$$