Calculus 1 – Spring 2019 Section 2
HW7

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Remark. The due date is April 3rd, 2019.

1 Review

1.1 Exercise. Apply the following functions on the number $x \in \mathbb{R}$ (or a subset of $\mathbb{R}$ if need be to make sense). E.g. if the function is $\sin \circ \log$, your answer should be $\sin(\log(x))$.

1. $\cosh \circ \sin \circ \exp \circ \log$.
2. $\left(\frac{\sin}{\cos}\right)^2$.
3. $\sin \circ \sin$.
4. $\sin \sin$.
5. $\frac{1}{\cos} \circ \arccos$.

1.2 Exercise. Find the slope of the line in $\mathbb{R}^2$ passing through $(5,6) \in \mathbb{R}^2$ and tangent to the circle centered at $(0,0) \in \mathbb{R}^2$ with radius 2.

1.3 Exercise. Make a sketch of the graph of $\mathbb{R} \ni x \mapsto x^2 + \exp(x) \in \mathbb{R}$ by following the steps below:

1. Find out where it's positive and where it's negative.
2. Find out where it passes the horizontal axis.
3. Find out what happens at $\pm\infty$.
4. Study the derivative to find out where it is:
   (a) increasing / decreasing.
   (b) attains a global maximum / minimum.

1.4 Exercise. [Koenigsberger] Calculate the *sequential* limit

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt[n]{n} - 1\right) .$$

1.5 Exercise. [Koenigsberger] Define $\alpha := 10^{10^{10^{10^{10^{10^{10^{10}}}}}}}$ (a really big number). Define the following three sequences $a, b, c : \mathbb{N} \to \mathbb{R}$: For any $n \in \mathbb{N}$, they are given by the formulæ

$$\begin{aligned} a(n) &:= \sqrt{n+\alpha} - \sqrt{n} \\ b(n) &:= \sqrt{n+\sqrt{n}} - \sqrt{n} \\ c(n) &:= \sqrt{n+\frac{n}{\alpha}} - \sqrt{n} \end{aligned}$$

1. Show that for all $n \in \mathbb{N}$ such that $n < \alpha^2$,
$$a(n) > b(n) > c(n) .$$

2. Show that

$$\lim a = 0 \,.$$

3. Show that

$$\lim b = \frac{1}{2} \,.$$

4. Show that

$$\lim c = \infty \,.$$

2 Ongoing lecture material

2.1 Exercise. In this exercise we study the *linear* approximation of sin near zero:

1. Find the value of sin at zero, and call it $\alpha \in \mathbb{R}$.
2. Find the value of the derivative of sin at zero, and call it $\beta \in \mathbb{R}$.
3. Define a *linear* function (i.e. a function whose graph's sketch looks like a straight line) $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) := \alpha + \beta x$$

4. Use the limit definition of the derivative

$$\sin'(0) \equiv \lim_{a \to 0} \frac{1}{a} (\sin(a) - \sin(0))$$

in order to determine that for any $\varepsilon > 0$ there is some threshold $\delta(\varepsilon) > 0$ such that if $|x| < \delta(\varepsilon)$ then sin is arbitrarily close to f in the following way:

$$\alpha + \beta x - \varepsilon |x| < \sin(x) < \alpha + \beta x + \varepsilon |x| \,.$$

Conclude that since ε itself can be made arbitrarily small, the term extra correction term $\varepsilon |x|$ can be made arbitrarily small compared to βx (i.e. $\frac{\varepsilon|x|}{\beta x}$ can be made arbitrarily close to zero), as long as $|x| < \delta(\varepsilon)$, so that it is in this sense that f approximates sin.
You may consult the lecture notes at Remark 8.5.

2.2 Exercise. Prove that the function $x^3 + x - 1 = 0$ for the unknown $x \in \mathbb{R}$ has precisely one real root using Rolle's theorem.

2.3 Exercise. A metal oil container is to be manufactured so that it could contain volume (measured in cubic meters) $v \in \mathbb{R}$ of oil. The cost of fabricating the container is proportional to the surface area of the container, so that minimizing the surface area of the container minimizes the cost of fabrication.

1. Assuming one wants a cylindrically-shaped container, find the optimal radius $r > 0$ and height $h > 0$ of the cylinder such that the cost to fabricate a container of volume v is minimal. You may use the fact that the volume of a cylinder is

$$V_{\text{cylinder}} = \pi r^2 h$$

and its surface area is

$$A_{\text{cylinder}} = 2\pi r h + 2 \cdot \pi r^2 \,.$$

2. Assuming one wants a conus-shaped container, find the optimal radius $r > 0$ and height $h > 0$ of the conus such that the cost to fabricate a container of volume v is minimal. You may use the fact that the volume of a conus is

$$V_{\text{conus}} = \frac{1}{3} \pi r^2 h$$

and its surface area is

$$A_{\text{conus}} = \pi r^2 + \pi r \sqrt{r^2 + h^2} \,.$$

Hint: In either shape, plug in $V_{\text{shape}} = v$ to solve for h. Then plug this h into A_{shape} to obtain a function of r alone (it will also depend on v, but v is fixed throughout). Now find the derivative of

$$(0, \infty) \ni r \quad \mapsto \quad A_{\text{shape}} \in (0, \infty)$$

and equate it to zero to get an equation for r_{optimal}. After having found r_{optimal}, go back and find from this h_{optimal} in terms of r_{optimal} and v.

2.4 Exercise. Find the point $(a, b) \in \mathbb{R}^2$ on the parabola defined by the set of all points $(x, y) \in \mathbb{R}^2$ obeying the equation

$$y^2 = 2x$$

such that the distance between (a, b) and $(1, 4) \in \mathbb{R}^2$ is minimal.

2.5 Exercise. At which points on the sketch of the function $\mathbb{R} \ni x \mapsto 1 + 40x^3 - 3x^5 \in \mathbb{R}$ does the tangent line have the largest slope?