Calculus 1 – Spring 2019 Section 2
HW9 *Solutions*

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Remark. The due date is April 15th, 2019.

1 Review

This week there will be no review.

2 Ongoing lecture material

2.1 Exercise. Calculate the area *almost*-triangular shape formed by the following three curves:

1. The horizontal line interval $[0,2]$ at height zero (i.e. *on* the horizontal axis).
2. The graph of $[0,1] \ni x \mapsto x^2$.
3. The graph of $[1,2] \ni x \mapsto (x-2)^2$.

You may find Theorem 9.25 and Remark 9.34 in the lecture notes useful.

Solution. Let us define $f : [0,2] \to \mathbb{R}$ by

$$x \quad \mapsto \quad \begin{cases} x^2 & x \in [0,1] \\ (x-2)^2 & x \in [1,2] \end{cases} .$$

Note this function is continuous (it attains the value 1 at the point of stitching 1 if we use either version of the piecewise formula). Hence by Theorem 9.16 it is integrable. The question is asking us (in words, if we use our geometric interpretation of the integral as the area underneath a curve of a function) to calculate

$$\int_0^2 f \,.$$

Using Theorem 9.25 we separate the integration over $[0,2]$ to two separate integrations, over $[0,1]$ and $[1,2]$ respectively, according to how the piecewise formula is stitched. We then find

$$\int_0^2 f \quad = \quad \int_0^1 x^2 \mathrm{d}x + \int_0^1 (x-2)^2 \,\mathrm{d}x \,.$$

At this point we may use the laws from Remark 9.34 (namely the power law) on the first integral to get

$$\begin{aligned} \int_0^1 x^2 \mathrm{d}x \quad &= \quad \frac{1}{3}x^3 \bigg|_{x=0}^{x=1} \\ &= \quad \frac{1}{3} \,. \end{aligned}$$

To deal with the second integral, let us perform the change of variables (Theorem 9.35–make sure its conditions are satisfied) by defining $\varphi(x) := x-2$ and $\varphi'(x) = 1$ for all $x \in [1,2]$. Then

$$(x-2)^2 \quad = \quad \varphi(x)^2 \varphi'(x)$$

so that Theorem 9.35 implies

$$\begin{aligned}\int_1^2 (x-2)^2 \, \mathrm{d}x &= \int_1^2 \varphi(x)^2 \varphi'(x) \, \mathrm{d}x \\ &= \int_{\varphi(1)}^{\varphi(2)} x^2 \mathrm{d}x \\ &= \int_{-1}^{0} x^2 \mathrm{d}x \,.\end{aligned}$$

We then use Remark 9.34 again (the power law) to get

$$\begin{aligned}\int_{-1}^{0} x^2 \mathrm{d}x &= \frac{1}{3} x^3 \Big|_{x=-1}^{x=0} \\ &= \frac{1}{3}\left((0)^3 - (-1)^3\right) \\ &= \frac{1}{3} \,.\end{aligned}$$

The final answer is thus, apparently,

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \,.$$

2.2 Exercise. A car was driving for two hours with instantaneous speed given by the function of time (measured in miles per hour)

$$\begin{aligned} s : [0,2] &\to \mathbb{R} \\ t &\mapsto \sqrt{t} \,. \end{aligned}$$

Find the total distance travelled by the car. Recall velocity is the derivative in time of distance, so to find the total distance you must integrate the speed. You may find Remark 9.34 item (1) useful.

Solution. Let us define the distance function

$$d : [0,2] \to \mathbb{R}$$

which tells us the distance attained (in miles) as a function of time. We define it so that $d(0) = 0$ (the distance at time zero better be zero). The hint is telling us that the following relationship exists

$$d' = s \,.$$

We have been given a formula for s, but apparently what we want to know is d (which will give us $d(2)$, the distance attained after two hours, which is what we are really interested in). For this, we can *integrate both sides of the above equation*. What this means is that if for two functions f, g we have

$$f(x) = g(x) \qquad (\text{ for all } x)$$

then of course

$$\int_a^b f(x) \, \mathrm{d}x = \int_a^b g(x) \, \mathrm{d}x \,.$$

Hence we have, by Theorem 9.32, using our convention $d(0) = 0$, and finally the equation $d' = s$,

$$\begin{aligned} d(2) &= d(2) - d(0) \\ &= \int_0^2 d' \\ &= \int_0^2 s \\ &= \int_0^2 \sqrt{t} \mathrm{d}t \,, \end{aligned}$$

where in the last line we used the given formula for s. At this point we can use Remark 9.34 to find

$$\begin{aligned}\int_0^2 \sqrt{t}\mathrm{d}t &= \left.\frac{1}{\frac{1}{2}+1}t^{\frac{1}{2}+1}\right|_{t=0}^{t=2}\\ &= \frac{2}{3}\left(2^{\frac{3}{2}}-0^{\frac{3}{2}}\right)\\ &= \frac{2^{\frac{5}{2}}}{3}\,.\end{aligned}$$

So the final answer is that the distance attained after two hours is $\frac{2^{\frac{5}{2}}}{3}$ miles (the car is apparently traveling *very* slowly).

2.3 Exercise. We know using Remark 9.34 that

$$\int_a^b \cos = \sin|_a^b$$

so in particular

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos &= \left(\sin\left(\frac{\pi}{2}\right)-\sin(0)\right)\\ &= 1\,.\end{aligned}$$

Now we want to evaluate this explicitly from Definition 9.8, i.e., take the limits of the lower and upper Darboux sums.

1. Write down $\overline{S}_0^{\frac{\pi}{2}}(\cos, N)$ and $\underline{S}_0^{\frac{\pi}{2}}(\cos, N)$ approximating

$$\int_0^{\frac{\pi}{2}} \cos$$

   at some finite N. Use the fact that cos is strictly monotone decreasing on $\left[0, \frac{\pi}{2}\right]$ in order to simplify the supremums and infimums to the left and right endpoints of each sub-interval, respectively. Simplify as much as you can.

2. Use the so-called *Dirichlet kernel formula*

$$1+2\sum_{k=1}^{L}\cos(k\theta) = \frac{\sin\left(\left(L+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} \qquad (\theta\in\mathbb{R}, L\in\mathbb{N})$$

   on the expressions you found for $\overline{S}_0^{\frac{\pi}{2}}(\cos, N)$ and $\underline{S}_0^{\frac{\pi}{2}}(\cos, N)$ in order to get rid of the sums. Simplify as much as you can.

3. Identify two types of terms in your expressions: Terms that behave like $\frac{1}{N}$ and others that can be brought to the form

$$\frac{\sin\left(\text{something } + \text{something}\frac{1}{N}\right)}{\operatorname{sinc}\left(\text{something}\frac{1}{N}\right)}$$

   Now use the fact that $\operatorname{sinc}(\alpha)\to 1$ as $\alpha\to 0$ (see Example 6.32) and the continuity of sin (so as to push the limit through) to conclude the result that both your expressions converge, in the limit $N\to\infty$, to 1.

Solution. We follow the suggested steps:

1. Since cos is monotone decreasing on $\left[0, \frac{\pi}{2}\right]$, we have for any two numbers c, d such that $0\le c<d\le\frac{\pi}{2}$, that

$$\begin{aligned}\sup(\{\cos(x)\mid x\in[c,d]\}) &= \cos(c)\\ \inf(\{\cos(x)\mid x\in[c,d]\}) &= \cos(d)\,.\end{aligned}$$

   Hence

$$\begin{aligned}\sup\left(\left\{\cos(x)\in\mathbb{R}\,\middle|\, x\in\left[n\frac{\frac{\pi}{2}}{N},(n+1)\frac{\frac{\pi}{2}}{N}\right]\right\}\right) &= \cos\left(\frac{\pi n}{2N}\right)\\ \inf\left(\left\{\cos(x)\in\mathbb{R}\,\middle|\, x\in\left[n\frac{\frac{\pi}{2}}{N},(n+1)\frac{\frac{\pi}{2}}{N}\right]\right\}\right) &= \cos\left(\frac{\pi(n+1)}{2N}\right)\,.\end{aligned}$$

We hence can simplify the lower and upper Darboux sums (Definition 9.8) as follows:

$$\begin{aligned}
\overline{S}_0^{\frac{\pi}{2}}(\cos, N) &= \frac{\frac{\pi}{2}}{N}\sum_{n=0}^{N-1}\cos\left(\frac{\pi n}{2N}\right)\\
&= \frac{\pi}{2N}\sum_{n=0}^{N-1}\cos\left(\frac{\pi n}{2N}\right)
\end{aligned}$$

and

$$\begin{aligned}
\underline{S}_0^{\frac{\pi}{2}}(\cos, N) &= \frac{\frac{\pi}{2}}{N}\sum_{n=0}^{N-1}\cos\left(\frac{\pi(n+1)}{2N}\right)\\
&= \frac{\pi}{2N}\sum_{n=0}^{N-1}\cos\left(\frac{\pi(n+1)}{2N}\right).
\end{aligned}$$

2. Let us first understand $\overline{S}_0^{\frac{\pi}{2}}(\cos, N)$, which involves $\sum_{n=0}^{N-1}\cos\left(\frac{\pi n}{2N}\right)$. Let us re-write

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos\left(\frac{\pi n}{2N}\right) &= \cos\left(\frac{\pi 0}{2N}\right)+\sum_{n=1}^{N-1}\cos\left(\frac{\pi n}{2N}\right)\\
&= 1+\sum_{n=1}^{N-1}\cos\left(\frac{\pi n}{2N}\right)
\end{aligned}$$

We can now use the Dirichlet kernel formula

$$\sum_{k=1}^{L}\cos(k\theta) = \frac{1}{2}\left(\frac{\sin\left(\left(L+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}-1\right)$$

with $L = N-1$ and $\theta = \frac{\pi}{2N}$ to get

$$\begin{aligned}
\sum_{n=1}^{N-1}\cos\left(\frac{\pi n}{2N}\right) &= \frac{1}{2}\left(\frac{\sin\left(\left(N-1+\frac{1}{2}\right)\frac{\pi}{2N}\right)}{\sin\left(\frac{1}{2}\frac{\pi}{2N}\right)}-1\right)\\
&= \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{2\sin\left(\frac{\pi}{4N}\right)}-\frac{1}{2}.
\end{aligned}$$

All together we collect our result to get

$$\begin{aligned}
\overline{S}_0^{\frac{\pi}{2}}(\cos, N) &= \frac{\pi}{2N}\left(1+\sum_{n=1}^{N-1}\cos\left(\frac{\pi n}{2N}\right)\right)\\
&= \frac{\pi}{2N}\left(1+\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{2\sin\left(\frac{\pi}{4N}\right)}-\frac{1}{2}\right)\\
&= \frac{\pi}{4N}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\sin\left(\frac{\pi}{4N}\right)}+\frac{\pi}{4N}\\
&= \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\frac{\sin\left(\frac{\pi}{4N}\right)}{\frac{\pi}{4N}}}+\frac{\pi}{4N}\\
&= \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\operatorname{sinc}\left(\frac{\pi}{4N}\right)}+\frac{\pi}{4N}.
\end{aligned}$$

Now we deal with the lower sum. Using a change in the summation index $n \mapsto n+1$ and then the Dirichlet kernel again, we get:

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos\left(\frac{\pi(n+1)}{2N}\right) &= \sum_{n=1}^{N}\cos\left(\frac{\pi n}{2N}\right)\\
&= \frac{1}{2}\left(\frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{\pi}{2N}\right)}{\sin\left(\frac{1}{2}\frac{\pi}{2N}\right)}-1\right)\\
&= \frac{\sin\left(\frac{\pi}{2}+\frac{\pi}{24}\right)}{2\sin\left(\frac{\pi}{4N}\right)}-\frac{1}{2}
\end{aligned}$$

and hence

$$\begin{aligned}
\underline{S}_{0}^{\frac{\pi}{2}}(\cos, N) &= \frac{\pi}{2N}\sum_{n=0}^{N-1}\cos\left(\frac{\pi(n+1)}{2N}\right) \\
&= \frac{\sin\left(\frac{\pi}{2}+\frac{\pi}{24}\right)}{\operatorname{sinc}\left(\frac{\pi}{4N}\right)} - \frac{\pi}{4N}\,.
\end{aligned}$$

3. We are interested in the limit $N \to \infty$, so

$$\begin{aligned}
\lim_{N\to\infty}\overline{S}_{0}^{\frac{\pi}{2}}(\cos, N) &= \lim_{N\to\infty}\left(\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\operatorname{sinc}\left(\frac{\pi}{4N}\right)} + \frac{\pi}{4N}\right) \\
&\quad \text{(Algebra of limits)} \\
&= \left(\lim_{N\to\infty}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\operatorname{sinc}\left(\frac{\pi}{4N}\right)}\right) + \underbrace{\left(\lim_{N\to\infty}\frac{\pi}{4N}\right)}_{=0} \\
&\quad \text{(Algebra of limits)} \\
&= \frac{\lim_{N\to\infty}\sin\left(\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\lim_{N\to\infty}\operatorname{sinc}\left(\frac{\pi}{4N}\right)} \\
&\quad \text{(sinc and sin are continuous, so we can push the limits through)} \\
&= \frac{\sin\left(\lim_{N\to\infty}\frac{\pi}{2}-\frac{\pi}{4N}\right)}{\operatorname{sinc}\left(\lim_{N\to\infty}\frac{\pi}{4N}\right)} \\
&= \frac{\sin\left(\frac{\pi}{2}\right)}{\operatorname{sinc}(0)} \\
&\quad \text{(Use the fact that } \operatorname{sinc}(0)=1\text{)} \\
&= 1\,.
\end{aligned}$$

We get the same result for the limit of the lower sum, as the only difference is in the sign of the terms that anyway have zero limits.

2.4 Exercise. Consider the function from Example 9.18. Prove that its integral is zero by calculating explicitly the upper and lower Darboux sums and taking the limit $N \to \infty$.

Solution. The example from Example 9.18 was

$$\begin{aligned}
f : [0,1] &\to \mathbb{R} \\
x &\mapsto \begin{cases} 5 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}\,.
\end{aligned}$$

The lower sums will always be zero. Indeed, we have

$$\underline{S}_{0}^{1}(f, N) \equiv \frac{1}{N}\sum_{n=0}^{N-1}\inf\left(\left\{ f(x)\in\mathbb{R} \,\middle|\, x\in\left[\frac{n}{N},\frac{n+1}{N}\right]\right\}\right)$$

by definition, $f(x) = 0$ unless $x = \frac{1}{2}$. Hence, for fixed N, and for a fixed $n = 0, \dots, N-1$,

$$\inf\left(\left\{ f(x)\in\mathbb{R} \,\middle|\, x\in\left[\frac{n}{N},\frac{n+1}{N}\right]\right\}\right) = 0$$

if $\frac{1}{2} \notin \left[\frac{n}{N}, \frac{n+1}{N}\right]$ since f is just always zero on such an interval. If $\frac{1}{2} \in \left[\frac{n}{N}, \frac{n+1}{N}\right]$, since $\left[\frac{n}{N}, \frac{n+1}{N}\right] \neq \left\{\frac{1}{2}\right\}$ (for that to happen we would need to have $\frac{1}{2} = \frac{n}{N} = \frac{n+1}{N}$ which is impossible, since it implies $\frac{1}{N} = 0$ which is never the case!), there are *more* points other than $\frac{1}{2}$ in $\left[\frac{n}{N}, \frac{n+1}{N}\right]$ such that on those points, f is zero. Hence the infimum over such an interval that contains $\frac{1}{2}$ but also other points, is still zero. We learn that

$$\begin{aligned}
\underline{S}_{0}^{1}(f, N) &= \frac{1}{N}\sum_{n=0}^{N-1} 0 \\
&= 0\,.
\end{aligned}$$

The limit of the constant sequence zero is of course zero.

The upper sums are defined as

$$\overline{S}_0^1(f,N) \equiv \frac{1}{N}\sum_{n=0}^{N-1}\sup\left(\left\{\, f(x)\in\mathbb{R} \,\middle|\, x\in\left[\frac{n}{N},\frac{n+1}{N}\right]\,\right\}\right).$$

Here, again, if for fixed N, and for a fixed $n=0,\dots,N-1$, $\frac{1}{2}\notin\left[\frac{n}{N},\frac{n+1}{N}\right]$, $\sup\left(\left\{\, f(x)\in\mathbb{R} \mid x\in\left[\frac{n}{N},\frac{n+1}{N}\right]\,\right\}\right)=0$. However, unlike before, now we have for that special $n_\star$ (there must be one!, since $\frac{1}{2}$ must be in at least one interval, if not two, in case $\frac{n}{N}=\frac{1}{2}$ by chance) such that $\frac{1}{2}\in\left[\frac{n}{N},\frac{n+1}{N}\right]$,

$$\sup\left(\left\{\, f(x)\in\mathbb{R} \,\middle|\, x\in\left[\frac{n}{N},\frac{n+1}{N}\right]\,\right\}\right) = 5$$

Hence we learn that (the sum is zero on all other n's which aren't special) (the bound is in case there are two respectively one special $n_\star$, depending on whether $\frac{n}{N}=\frac{1}{2}$ by chance)

$$\frac{1}{N}\times 5 \le \overline{S}_0^1(f,N) \le \frac{2}{N}\times 5$$

so that using the squeeze theorem we find

$$\lim_{N\to\infty}\overline{S}_0^1(f,N) = 0\,.$$

2.5 Exercise. For $b>1$, give upper and lower bounds (which depend on b)

$$\int_1^b \log\cdot\sin$$

using Theorem 9.24 and the following bounds (see Claim 10.3):

$$\begin{aligned}\operatorname{im}(\sin) &\subseteq [-1,1]\\ \log(x) &\le x-1 \qquad (x>0)\\ \log(x) &\ge 1-\frac{1}{x} \qquad (x>0)\,.\end{aligned}$$

We note that while log and sin are each continuous, which means that the product log sin is also continuous, and hence by Theorem 9.16 it is clear that it is integrable. However, it is very hard to write down an explicit formula for this integral (though we can integrate each separately) which is why it is useful to have bounds on the integral, in order to understand worst or best case scenarios for the result.

Solution. This exercise wasn't fully thought out and in hindsight I should have only asked for an upper bound. Let us derive this upper bound:

$$\begin{aligned}\int_1^b \log\cdot\sin &\le \left|\int_1^b \log\cdot\sin\right|\\ &\quad\text{(Theorem 9.29)}\\ &\le \int_1^b |\log|\,|\sin|\\ &\quad(|\sin|\le 1 \text{ and Theorem 9.24})\\ &\le \int_1^b |\log|\end{aligned}$$

Now since we are evaluating log on $[1,b]$, it will always be positive so that the absolute value is redundant. Then we get

$$\begin{aligned}\int_1^b |\log| &= \int_1^b \log \\ &\quad \text{(Use the given upper bound and Theorem 9.24)} \\ &\leq \int_1^b (x-1)\,\mathrm{d}x \\ &\quad \text{(Integrate using the power law of Remark 9.34)} \\ &= \left(\frac{1}{2}x^2 - x\right)\Bigg|_{x=1}^{x=b} \\ &= \frac{1}{2}b^2 - b - \left(\frac{1}{2}-1\right) \\ &= \frac{1}{2}\left(b\left(b-1\right)+1\right)\,.\end{aligned}$$

The final result is:

$$\int_1^b \log\sin \quad \leq \quad \frac{1}{2}\left(b\left(b-1\right)+1\right)\,.$$

For the sake of completeness we also give the lower bound, though it is very far from the material of the class! (so only waste time reading it if you are very very bored).

We have

$$\log\sin \quad = \quad \log\left(-\cos'\right)$$

so that using integration by parts (Theorem 9.41) we get

$$\begin{aligned}\int_1^b \log\sin &= -\int_1^b \log\cos' \\ &\quad \left(\log'(x) = \frac{1}{x}\right) \\ &= -\log\cos|_1^b + \int_1^b \frac{1}{x}\cos(x)\,\mathrm{d}x \\ &= -\log(b)\cos(b) + \int_1^b \frac{1}{x}\cos(x)\,\mathrm{d}x\end{aligned}$$

Let us assume for simplicity that $b = \frac{\pi}{2} + 2\pi m$ for some $m \in \mathbb{N}$ (if this is not the case one has to take into account more pieces which anyway go to zero as $b \to \infty$). In this case, then, actually $\cos(b) = 0$. Note that cos changes signs repeatedly along $[1,b]$: it starts off being positive, at $\frac{\pi}{2} \approx 1.5$ it is zero, then it becomes negative until $\frac{3\pi}{2}$ and then it is positive again until $\frac{\pi}{2} + 2\pi$, and on and on. Hence we can separate the integral using Theorem 9.25 as

$$\begin{aligned}\int_1^b \frac{1}{x}\cos(x)\,\mathrm{d}x &= \int_1^{\frac{\pi}{2}+2\pi m} \frac{1}{x}\cos(x)\,\mathrm{d}x \\ &= \int_1^{\frac{\pi}{2}} \frac{1}{x}\cos(x)\,\mathrm{d}x + \sum_{n=1}^{m} \underbrace{\int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi} \frac{1}{x}\cos(x)\,\mathrm{d}x}_{\text{negative}} + \underbrace{\int_{\frac{3\pi}{2}+2(n-1)\pi}^{\frac{\pi}{2}+2n\pi} \frac{1}{x}\cos(x)\,\mathrm{d}x}_{\text{positive}}\end{aligned}$$

Let us estimate the first term, $\int_1^{\frac{\pi}{2}} \frac{1}{x}\cos(x)\,\mathrm{d}x$. Since $x \leq \frac{\pi}{2}$, we have $\frac{1}{x} \geq \frac{2}{\pi}$ so that

$$\begin{aligned}\int_1^{\frac{\pi}{2}} \frac{1}{x}\cos(x)\,\mathrm{d}x &\geq \int_1^{\frac{\pi}{2}} \frac{2}{\pi}\cos(x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(1)\right) \\ &= \frac{2}{\pi}\left(1-\sin(1)\right) \\ &\approx 0.1\,.\end{aligned}$$

For the other terms, let us make a change of variable in the integral $\int_{\frac{3\pi}{2}+2(n-1)\pi}^{\frac{\pi}{2}+2n\pi}\frac{1}{x}\cos\left(x\right)\mathrm{d}x$ so as to shift things by π:

$$\int_{\frac{3\pi}{2}+2(n-1)\pi}^{\frac{\pi}{2}+2n\pi}\frac{1}{x}\cos\left(x\right)\mathrm{d}x \quad = \quad \int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi}\frac{1}{x+\pi}\cos\left(x+\pi\right)\mathrm{d}x$$

but now, $\cos\left(x-\pi\right)=-\cos\left(x\right)$, so that

$$\int_{\frac{3\pi}{2}+2(n-1)\pi}^{\frac{\pi}{2}+2n\pi}\frac{1}{x}\cos\left(x\right)\mathrm{d}x \quad = \quad -\int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi}\frac{1}{x+\pi}\cos\left(x\right)\mathrm{d}x$$

and so the whole sum becomes

$$\begin{aligned}\sum_{n=1}^{m}\cdots &= \sum_{n=1}^{m}\underbrace{\int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi}\frac{1}{x}\cos\left(x\right)\mathrm{d}x}_{\text{negative}}-\underbrace{\int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi}\frac{1}{x+\pi}\cos\left(x\right)\mathrm{d}x}_{\text{negative}}\\ &= \sum_{n=1}^{m}\int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi}\left(\frac{1}{x}-\frac{1}{x+\pi}\right)\cos\left(x\right)\mathrm{d}x\\ &= \pi\sum_{n=1}^{m}\underbrace{\int_{\frac{\pi}{2}+2(n-1)\pi}^{\frac{3\pi}{2}+2(n-1)\pi}\frac{1}{x\left(x+\pi\right)}\cos\left(x\right)\mathrm{d}x}_{\text{negative}}\end{aligned}$$

Now that all terms have the same sign, it is easier to estimate them without worrying about cancellations (because we made the cancellations built into the sum). Thus, for example, since we are looking for a lower bound and these terms are negative, we can get away with an upper bound, so

$$\left|\frac{1}{x\left(x+\pi\right)}\cos\left(x\right)\right| \quad \leq \quad \frac{1}{x\left(x+\pi\right)}$$

and then $x+\pi\geq x$ so $\frac{1}{x+\pi}\leq\frac{1}{x}$ and hence

$$\frac{1}{x\left(x+\pi\right)} \quad \leq \quad \frac{1}{x^{2}}$$

and finally since within the integral, $x\geq\frac{\pi}{2}+2\left(n-1\right)\pi\geq n$, we find

$$\frac{1}{x^{2}} \quad \leq \quad \frac{1}{n^{2}}\,.$$

All in all we find

$$\left|\sum_{n=1}^{m}\cdots\right| \quad \leq \quad \pi^{2}\sum_{n=1}^{m}\frac{1}{n^{2}}\,.$$

The final conclusion is:

$$\int_{1}^{\frac{\pi}{2}+2\pi m}\log\sin \quad \geq \quad \frac{2}{\pi}\left(1-\sin\left(1\right)\right)-\pi^{2}\sum_{n=1}^{m}\frac{1}{n^{2}}\,.$$

Coincidentally, in the limit $b\to\infty$, (or $m\to\infty$), $\sum_{n=1}^{m}\frac{1}{n^{2}}=\frac{\pi^{2}}{6}$ so that we get

$$\begin{aligned}\int_{1}^{\infty}\log\sin &\geq \frac{2}{\pi}\left(1-\sin\left(1\right)\right)-\frac{\pi^{4}}{6}\\ &\approx -16.13\end{aligned}$$

Note that a compute approximation gives $\int_{1}^{\infty}\log\sin\approx-0.33$.

2.6 Exercise. [Paul] Use Theorem 9.35 and then Remark 9.34 to evaluate the following integrals (for some $a,b\in\mathbb{R}$ such that $a<b$)

1. $\int_{a}^{b}\left(1-\frac{1}{x}\right)\cos\left(x-\log\left(x\right)\right)\mathrm{d}x$.

2. $\int_a^b 3\left(8x-1\right)\mathrm{e}^{4x^2-x}\mathrm{d}x$.

3. $\int_a^b x^2\left(3-10x^3\right)^4\mathrm{d}x$.

4. $\int_a^b \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x$.

5. $\int_a^b \sin\left(1-x\right)\left(2-\cos\left(1-x\right)\right)^4\mathrm{d}x$.

6. $\int_a^b \cos\left(3x\right)\left(\sin\left(3x\right)\right)^{10}\mathrm{d}x$.

7. $\int_a^b \frac{\left(3-\tan(4x)\right)^3}{\cos(4x)^2}\mathrm{d}x$.

8. $\int_a^b \frac{3}{5x+4}\mathrm{d}x$.

9. $\int_a^b \frac{3x}{5x^2+4}\mathrm{d}x$.

10. $\int_a^b \frac{3x}{\left(5x^2+4\right)^2}\mathrm{d}x$.

11. $\int_a^b \frac{3}{5x^2+4}\mathrm{d}x$.

12. $\int_a^b \frac{2x^3+1}{\left(x^4+2x\right)^3}\mathrm{d}x$.

13. $\int_a^b \frac{2x^3+1}{x^4+2x}\mathrm{d}x$.

14. $\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x$.

15. $\int \frac{1}{\sqrt{1-4x^2}}\mathrm{d}x$.

Solution. See the solution from where the exercises were taken: http://tutorial.math.lamar.edu/Classes/CalcI/SubstitutionRuleIndefinite.aspx.

2.7 Exercise. [Paul] Use Theorem 9.41 and then Remark 9.34 in order to evaluate the following integrals (for some $a, b \in \mathbb{R}$ such that $a < b$)

1. $\int_a^b x\mathrm{e}^{6x}\mathrm{d}x$.

2. $\int_a^b \left(3x+5\right)\cos\left(\frac{x}{4}\right)\mathrm{d}x$.

3. $\int_a^b x^2\sin\left(10x\right)\mathrm{d}x$.

4. $\int_a^b x\sqrt{x+1}\mathrm{d}x$.

5. $\int_a^b \log$ (hint: use the constant function $f\left(x\right)=1$ for all $x \in \mathbb{R}$).

6. $\int_a^b \mathrm{e}^x\cos\left(x\right)\mathrm{d}x$.

Solution. See the solution from where the exercises were taken: http://tutorial.math.lamar.edu/Classes/CalcII/IntegrationByParts.aspx.