Calculus 1 – Spring 2019 Section 2
HW9

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Remark. The due date is April 15th, 2019.

1 Review

This week there will be no review.

2 Ongoing lecture material

2.1 Exercise. Calculate the area *almost*-triangular shape formed by the following three curves:

1. The horizontal line interval $[0,2]$ at height zero (i.e. *on* the horizontal axis).
2. The graph of $[0,1] \ni x \mapsto x^2$.
3. The graph of $[1,2] \ni x \mapsto (x-2)^2$.

You may find Theorem 9.25 and Remark 9.34 in the lecture notes useful.

2.2 Exercise. A car was driving for two hours with instantaneous speed given by the function of time (measured in miles per hour)

$$\begin{aligned} s : [0,2] &\to \mathbb{R} \\ t &\mapsto \sqrt{t}\,. \end{aligned}$$

Find the total distance travelled by the car. Recall velocity is the derivative in time of distance, so to find the total distance you must integrate the speed. You may find Remark 9.34 item (1) useful.

2.3 Exercise. We know using Remark 9.34 that

$$\int_a^b \cos = \sin|_a^b$$

so in particular

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos &= \left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) \\ &= 1\,. \end{aligned}$$

Now we want to evaluate this explicitly from Definition 9.8, i.e., take the limits of the lower and upper Darboux sums.

1. Write down $\overline{S}_0^{\frac{\pi}{2}}(\cos, N)$ and $\underline{S}_0^{\frac{\pi}{2}}(\cos, N)$ approximating

   $$\int_0^{\frac{\pi}{2}} \cos$$

   at some finite N. Use the fact that cos is strictly monotone decreasing on $\left[0, \frac{\pi}{2}\right]$ in order to simplify the supremums and infimums to the left and right endpoints of each sub-interval, respectively. Simplify as much as you can.

2. Use the so-called *Dirichlet kernel formula*

$$1+2\sum_{k=1}^{L}\cos(k\theta) \quad = \quad \frac{\sin\left(\left(L+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} \qquad (\theta\in\mathbb{R}, L\in\mathbb{N})$$

on the expressions you found for $\overline{S}_0^{\frac{\pi}{2}}(\cos, N)$ and $\underline{S}_0^{\frac{\pi}{2}}(\cos, N)$ in order to get rid of the sums. Simplify as much as you can.

3. Identify two types of terms in your expressions: Terms that behave like $\frac{1}{N}$ and others that can be brought to the form

$$\frac{\sin\left(\text{something } +\text{something}\frac{1}{N}\right)}{\operatorname{sinc}\left(\text{something}\frac{1}{N}\right)}$$

Now use the fact that $\operatorname{sinc}(\alpha)\to 1$ as $\alpha\to 0$ (see Example 6.32) and the continuity of sin (so as to push the limit through) to conclude the result that both your expressions converge, in the limit $N\to\infty$, to 1.

2.4 Exercise. Consider the function from Example 9.18. Prove that its integral is zero by calculating explicitly the upper and lower Darboux sums and taking the limit $N\to\infty$.

2.5 Exercise. For $b>1$, give upper and lower bounds (which depend on b)

$$\int_1^b \log\cdot\sin$$

using Theorem 9.24 and the following bounds (see Claim 10.3):

$$\begin{aligned} \operatorname{im}(\sin) &\subseteq [-1,1] \\ \log(x) &\leq x-1 \qquad (x>0) \\ \log(x) &\geq 1-\frac{1}{x} \qquad (x>0)\ . \end{aligned}$$

We note that while log and sin are each continuous, which means that the product $\log\sin$ is also continuous, and hence by Theorem 9.16 it is clear that it is integrable. However, it is very hard to write down an explicit formula for this integral (though we can integrate each separately) which is why it is useful to have bounds on the integral, in order to understand worst or best case scenarios for the result.

2.6 Exercise. [Paul] Use Theorem 9.35 and then Remark 9.34 to evaluate the following integrals (for some $a,b\in\mathbb{R}$ such that $a<b$)

1. $\int_a^b \left(1-\frac{1}{x}\right)\cos(x-\log(x))\,\mathrm{d}x$.
2. $\int_a^b 3(8x-1)\,\mathrm{e}^{4x^2-x}\mathrm{d}x$.
3. $\int_a^b x^2\left(3-10x^3\right)^4\mathrm{d}x$.
4. $\int_a^b \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x$.
5. $\int_a^b \sin(1-x)\left(2-\cos(1-x)\right)^4\mathrm{d}x$.
6. $\int_a^b \cos(3x)\left(\sin(3x)\right)^{10}\mathrm{d}x$.
7. $\int_a^b \frac{(3-\tan(4x))^3}{\cos(4x)^2}\mathrm{d}x$.
8. $\int_a^b \frac{3}{5x+4}\mathrm{d}x$.
9. $\int_a^b \frac{3x}{5x^2+4}\mathrm{d}x$.
10. $\int_a^b \frac{3x}{(5x^2+4)^2}\mathrm{d}x$.
11. $\int_a^b \frac{3}{5x^2+4}\mathrm{d}x$.
12. $\int_a^b \frac{2x^3+1}{(x^4+2x)^3}\mathrm{d}x$.

13. $\int_a^b \frac{2x^3+1}{x^4+2x} \mathrm{d}x$.

14. $\int \frac{x}{\sqrt{1-4x^2}} \mathrm{d}x$.

15. $\int \frac{1}{\sqrt{1-4x^2}} \mathrm{d}x$.

2.7 Exercise. [Paul] Use Theorem 9.41 and then Remark 9.34 in order to evaluate the following integrals (for some $a, b \in \mathbb{R}$ such that $a < b$)

1. $\int_a^b x\mathrm{e}^{6x} \mathrm{d}x$.

2. $\int_a^b (3x+5) \cos\left(\frac{x}{4}\right) \mathrm{d}x$.

3. $\int_a^b x^2 \sin(10x) \mathrm{d}x$.

4. $\int_a^b x\sqrt{x+1} \mathrm{d}x$.

5. $\int_a^b \log$ (hint: use the constant function $f(x) = 1$ for all $x \in \mathbb{R}$).

6. $\int_a^b \mathrm{e}^x \cos(x) \mathrm{d}x$.